Maxwell's Equations and Light

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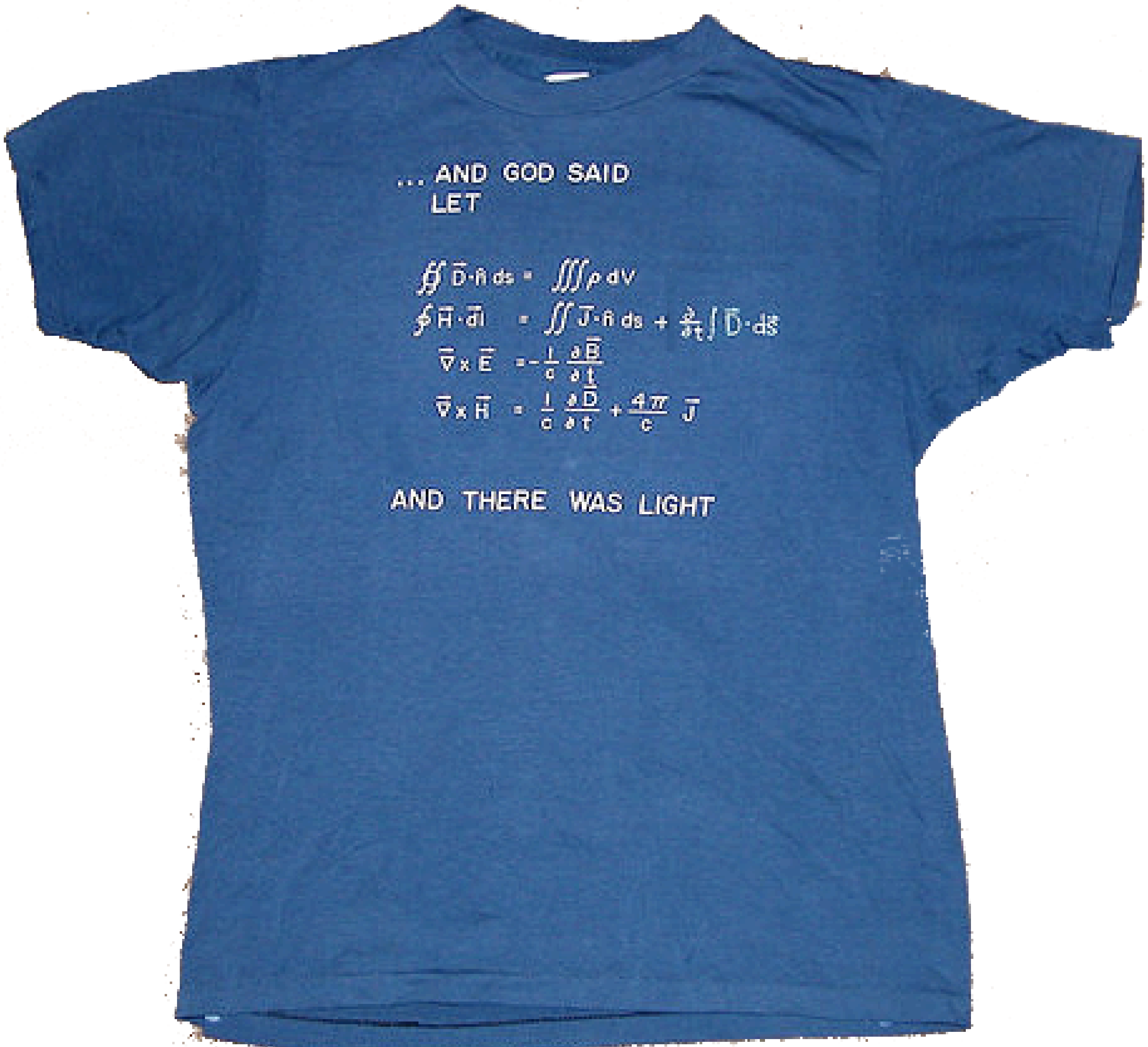
Physics 158

Peter Beyersdorf

Class Outline

- Maxwell's Equations
- The Wave Equation
- Light versus other EM waves

Maxwell's Equations



How do Maxwell's equations describe the propagation of Light?

Maxwell's Equations

Consider Maxwell's equations in differential form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$$ Gauss' law (for electricity)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's law

$$\vec{\nabla} \cdot \vec{B} = 0$$ Gauss' law (for magnetism)

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$ Ampere's law

What do each of these mean?

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Gauss' Law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$$

- Electrical charges are the source of the electric field
- The strength of the electric field depends on the material it is in (via ϵ)
- ϵ can be determined via measurements on a parallel plate capacitor filled with a given material using the equation $C=\epsilon A/d$

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Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- The Curl of the electric field is caused by changing magnetic fields
- A changing magnetic field can produce electric fields with field lines that close on themselves

Gauss' Law for Magnetism

$$\vec{\nabla} \cdot \vec{B} = 0$$

- There are no source of magnetic fields
  - No magnetic monopoles
  - Magnetic field lines can only circulate

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Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

- The Curl of the magnetic field is caused by current of charged particles (J) or of the field they produce (dE/dt)

- A changing electric field can produce magnetic fields (with field lines that close on themselves)

- The strength of the field depends on the material (via μ - in common materials μ=μo)

- μ can be measured by using the Biot-Savart law

$$d\vec{B} = \frac{\mu I d\vec{L} \times \hat{r}}{4\pi r^2}$$

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Waves and Maxwell's Equations

- A charged particle is a source of an electric field
- When that particle moves it changes the (spatial distribution of) the electric field
- When the electric field changes it produces a circulating magnetic field
- If the particle accelerates this circulating magnetic field will change
- A changing magnetic field produces a circulating electric field

Derivation of the Wave Equation

Starting with Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

take the curl of both sides

$$\begin{aligned} \vec{\nabla} \times \vec{\nabla} \times \vec{E} &= \vec{\nabla} \times (-\partial \vec{B}/\partial t) \\ &= -\partial(\vec{\nabla} \times \vec{B})/\partial t \end{aligned}$$

use vector calculus relationship to get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{E} = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t}$$

Use Ampere's law (in free space where J=0) $\vec{\nabla} \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$

and Gauss' law (in free space where ρ=0) $\vec{\nabla} \cdot \vec{E} = 0$

$$\boxed{\nabla^2 \vec{E} = \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}}$$

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Spherical Solutions to the Wave Equation

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Consider solutions for E such that $\nabla^2 E$ and $\delta^2 E/dt^2$ are both proportional to E - allowing the two sides to differ only by a constant term.

$$\vec{E}(\vec{r}, t) = \frac{\vec{E}_0}{r} e^{i\left(\vec{k}\cdot\vec{r} \pm \omega t\right)}$$

is one such solution in spherical coordinates. Using the relationship for the Laplacian of a spherically symmetric function:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi)$$

Show that $\vec{E}(\vec{r}, t)$ given above is a solution to the wave equation

2.

Solutions to the Wave Equation

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$k^2 = \mu\epsilon\omega^2$$

Recall the meaning of k and ω (k=2π/λ, ω=2π/T) we can express this as

$$\frac{\lambda}{T} = \frac{1}{\sqrt{\mu\epsilon}}$$

Since λ is the distance travelled by the wave in one cycle, and T is the time to travel one cycle, λ/T is the velocity of the wave, which can be determined from electrostatics and magnetostatics!

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$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

2.

Speed of Light

In free-space where $\epsilon=\epsilon_0$ and $\mu=\mu_0$ the speed of light is **defined** to be $c \equiv 299792458$ m/s. In this sense any measurement of the speed of light in a vacuum is really a measurement of the length of a meter (the unit of time is also a defined quantity)

In material where $\epsilon=\kappa\epsilon_0$ and $\mu=\mu_r\mu_0$ the speed of light is $v=c/(\kappa\mu_r)^{1/2}$. We let $n \equiv (\kappa\mu_r)^{1/2}$ and call n the **index of refraction** for a material.

What is the physical interpretation of n?

If it is complex, what do the real and imaginary parts represent?

Index of Refraction

From our expression for the velocity of the wave $v=c/(\kappa\mu_r)^{1/2}$ we can substitute $n=(\kappa\mu_r)^{1/2}$ to get $v=c/n$

Thus n represents how much slower light travels in a material compared to free space.

Given the relation $c=v/n=\omega/k$. If a wave travels from free space into a material causing it to slow down, does ω change, or does k change (or both)?

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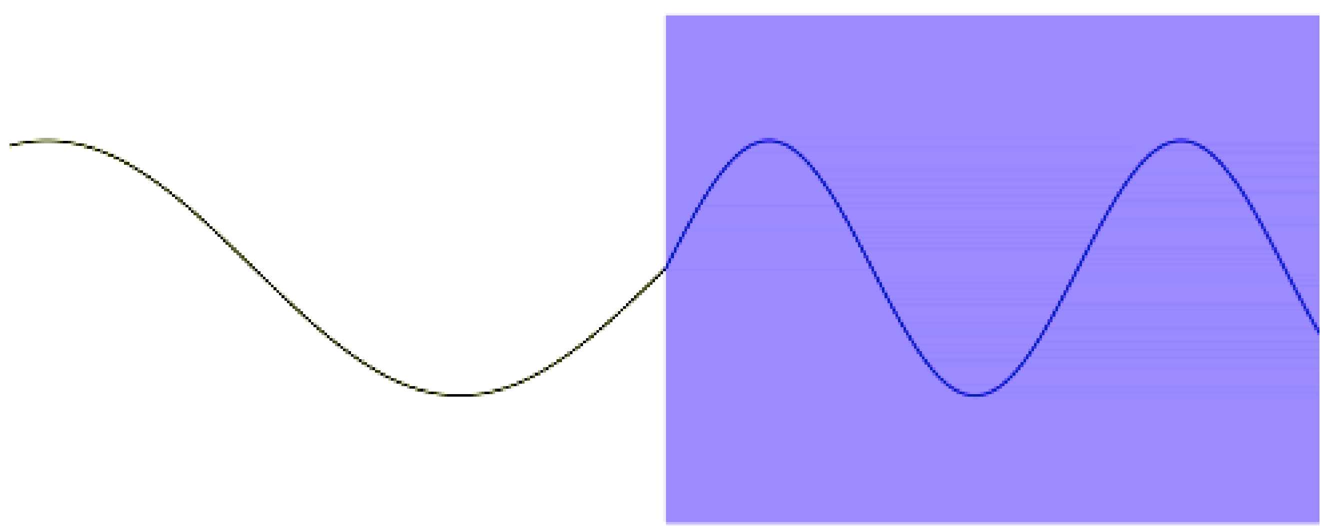
Index of Refraction

Consider a wave in free space entering a material. Doe the wavelength change, does the frequency change or both?



The frequency cannot change (or else the boundary would be discontinuous) so the wavelength (and hence k) must change so that $\lambda=\lambda_0/n$ and $k=nk_0$

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Index of Refraction

Going back to the solution to the wave equation, we can express it explicitly for propagation in a material with index of refraction n

$$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(n\vec{k}_0\cdot\vec{r}\pm\omega t\right)}$$

If n is complex such that n=n′+i n″

we have $$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(n'\vec{k}_0\cdot\vec{r}\pm\omega t\right)} e^{-n''\vec{k}_0\cdot\vec{r}}$$

and we see n″ is related to the absorption coefficient α used in Beer's law $I(x) = I_0 e^{-\alpha x}$ by $\alpha=2n''k_0$.

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Solutions to the Wave Equation

From our solution

$$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(\vec{k}\cdot\vec{r} \pm \omega t\right)}$$

and Gauss' law in free space (ρ=0)

$$\vec{\nabla} \cdot \vec{E} = 0$$

We find that since $\vec{E}(\vec{r},t)$ only has a spatial dependance on r its divergence

$$\vec{\nabla} \cdot \vec{\psi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \psi_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\ \psi_\phi) + \frac{1}{r\sin\phi}\frac{\partial \psi_\theta}{\partial \theta}$$

must be

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r(\vec{r},t)) = 0$$

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implying $E_r(\vec{r},t) = 0$ **meaning this is a transverse wave**2.

Describing Light

Since the Electric & Magnetic fields are both transverse to the direction of propagation, the direction of propagation can be found by the direction of their cross product.

Since the irradiance is proportional to the square of the eletric field, and the electric field is proportional to the magnetic field, the irradiance of a wave is proportional to the cross product of E and B

Poynting vector $\vec{S} \equiv c^2 \epsilon_0 \vec{E} \times \vec{B}$ gives the instantaneous flow of energy per unit area per unit time

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Describing Light

- Irradiance is the average power per unit area per unit time $I = < S >$
  it is often called the "intensity"

- The power passing through an area dA is

$$dP = I dA$$

- The average photon flux (number of photons per second) is

$$\Phi = \frac{P}{\hbar\omega}$$

- The rms variation in the photon flux over a time τ is

$$\sigma_n = \sqrt{n} = \sqrt{\Phi\tau} = \sqrt{\frac{P\tau}{\hbar\omega}}$$

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Summary

- Maxwell's equations result in electromagnetic waves that are transverse disturbances int he electric and magnetic fields moving at a speed of $v=(\epsilon\mu)^{-1/2}$

- The index of refaction, n, describes the relative speed of light through a material (real part) and the absorption in that material (imaginary part)

- Light can be described by various quantities

    - Poynting vector
    - irradiance
    - power
    - photon flux

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