silent "s"

Fresnel Equations

Tuesday, 9/12/2006

Physics 158

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Class Outline

- Boundary Conditions for EM waves
- Derivation of Fresnel Equations
- Consequences of Fresnel Equations
  - Amplitude of reflection coefficients
  - Phase shifts on reflection
  - Brewster's angle
  - Conservation of energy

Boundary Conditions

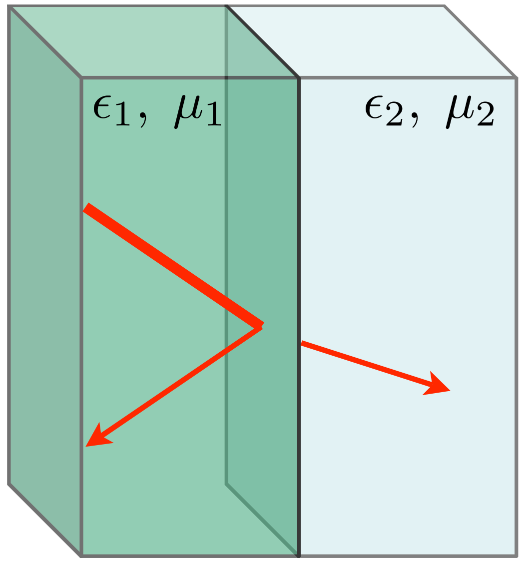
When an EM wave propagates across an interface, Maxwell's equations must be satisfied at the interface as well as in the bulk materials. The constraints necessary for this to occur are called the "boundary conditions"

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$



Boundary Conditions

Gauss' law can be used to find the boundary conditions on the component of the electric field that is perpendicular to the interface.

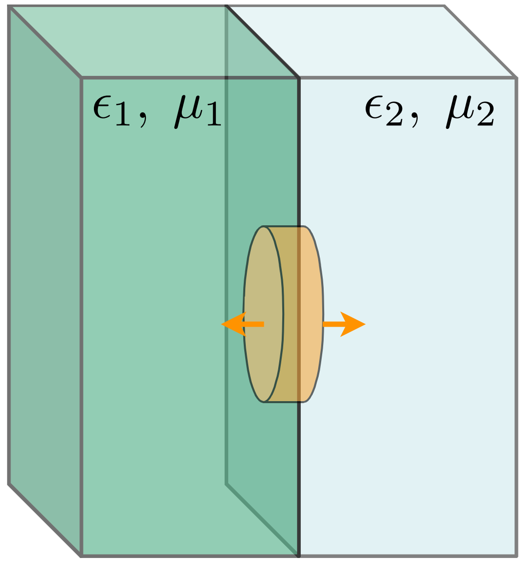
If the materials are dielectrics there will be no free charge on the surface (q=0)

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

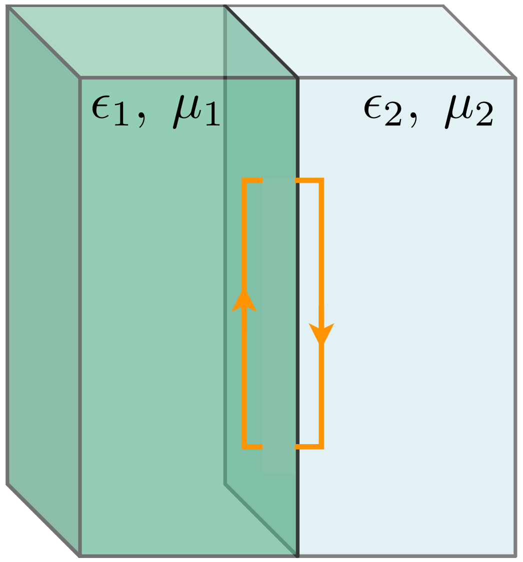
$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$



$$\epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = \sum q^{\nearrow 0} \quad \therefore \quad \epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$$

Boundary Conditions

Faraday's law can be applied at the interface. If the loop around which the electric field is computed is made to have an infintesimal area the right side will go to zero giving a relationship between the parallel components of the electric field



$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

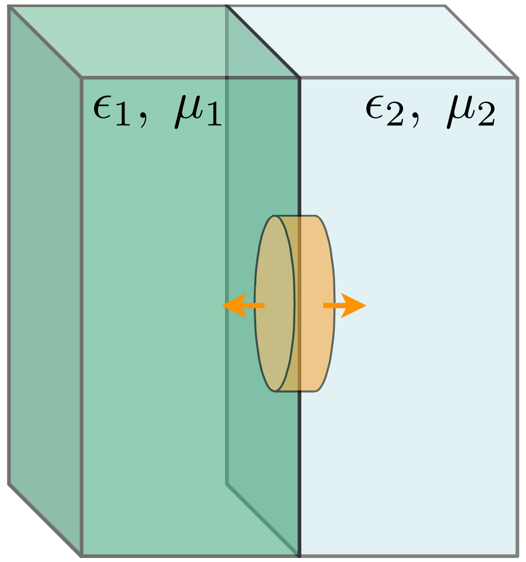
$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

$$E_{2\parallel} - E_{1\parallel} = -\frac{d}{dt}\int B \cdot dA \rightarrow 0 \qquad \therefore \qquad E_{1\parallel} = E_{2\parallel}$$

Boundary Conditions

Gauss' law for magnetism gives a relationship between the perpendicular components of the magnetic field at the interface



$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

$$B_{1\perp} A - B_{2\perp} A = 0 \qquad \therefore \qquad B_{1\perp} = B_{2\perp}$$

Boundary Conditions

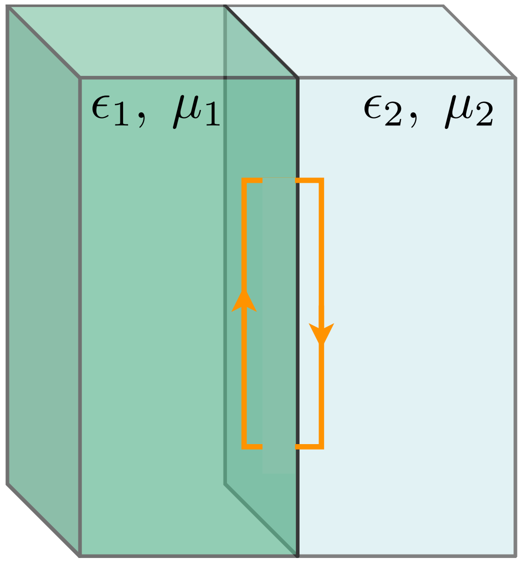
Ampere's law applied to a loop at the interface that has an infintesimal area gives a relationship between the parallel components of the magnetic field. (Note that in most common materials $\mu = \mu_o$)

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

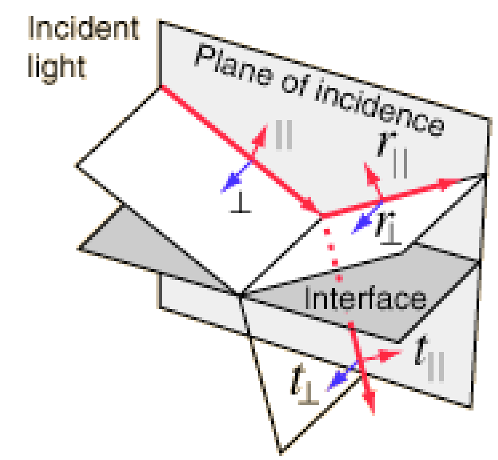


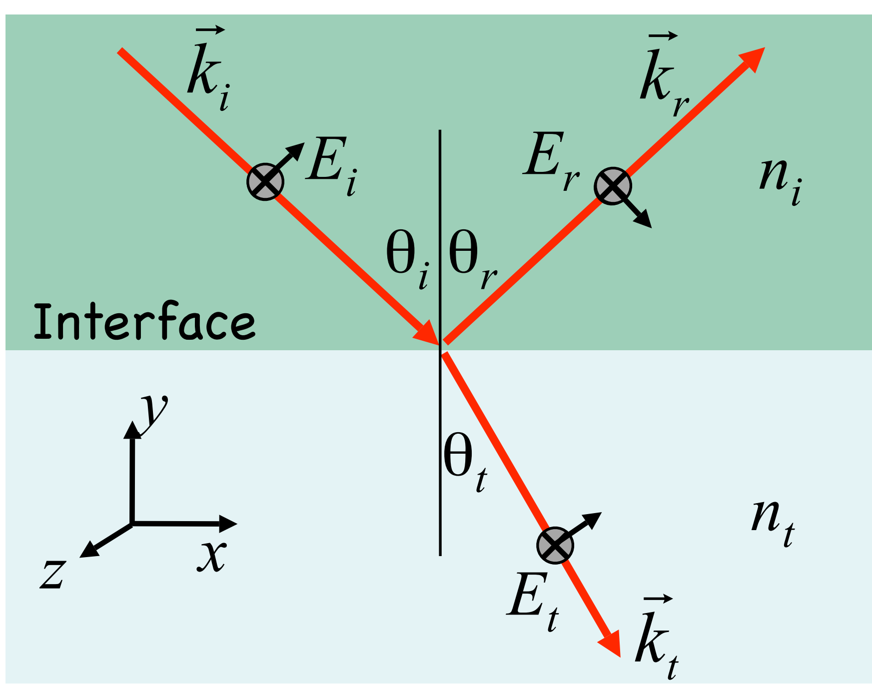
$$\frac{B_{1\parallel}}{\mu_1}L - \frac{B_{2\parallel}}{\mu_2}L = \int J \cdot \cancelto{0}{dA} + \frac{d}{dt}\int \epsilon E \cdot \cancelto{0}{dA} \quad \therefore$$

$$\frac{B_{1\parallel}}{\mu_1} = \frac{B_{2\parallel}}{\mu_2}$$

Reflection at a Boundary

The reflection and transmission coefficients at an interface can be found using the boundary conditions, but they depend on the polarization of the incident light





⊗ "s" polarization (senkrecht, aka TE or horizontal) has an E field that is perpendicular to the plane of incidence

↘ "p" polarization (parallel aka TM or vertical) has an E field that is parallel to the plane of incidence

$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$$

$$E_{1\|} = E_{2\|}$$

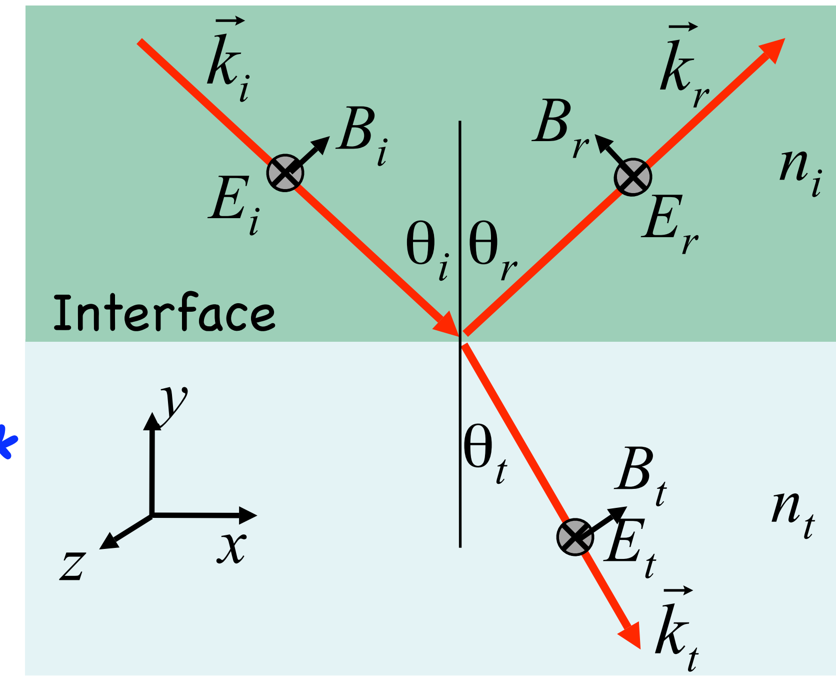
$$B_{1\perp} = B_{2\perp}$$

$$\frac{B_{1\|}}{\mu_1} = \frac{B_{2\|}}{\mu_2}$$

S-Polarization at a Boundary

The tangential electric field is continuous

$$\vec{E}_i(y=0,t) + \vec{E}_r(y=0,t) = \vec{E}_t(y=0,t)$$



The tangential magnetic field is continuous*

$$B_i(y=0,t)\cos\theta_i + B_r(y=0,t)\cos\theta_r$$
$$= B_t(y=0,t)\cos\theta_t$$

$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$$
$$E_{1\parallel} = E_{2\parallel}$$
$$B_{1\perp} = B_{2\perp}$$
$$\frac{B_{1\parallel}}{\mu_1} = \frac{B_{2\parallel}}{\mu_2}$$

Using $\theta_i = \theta_r$ and $B = nE/c$ and considering only the amplitude of the waves at the boundary

$$n_i\,(E_{0r} - E_{0i})\cos\theta_i = -n_t\,(E_{0r} + E_{0i})\cos\theta_t$$

*It's actually the tangential B/μ, but we're assuming $\mu = \mu_0$

S-Polarization at a Boundary

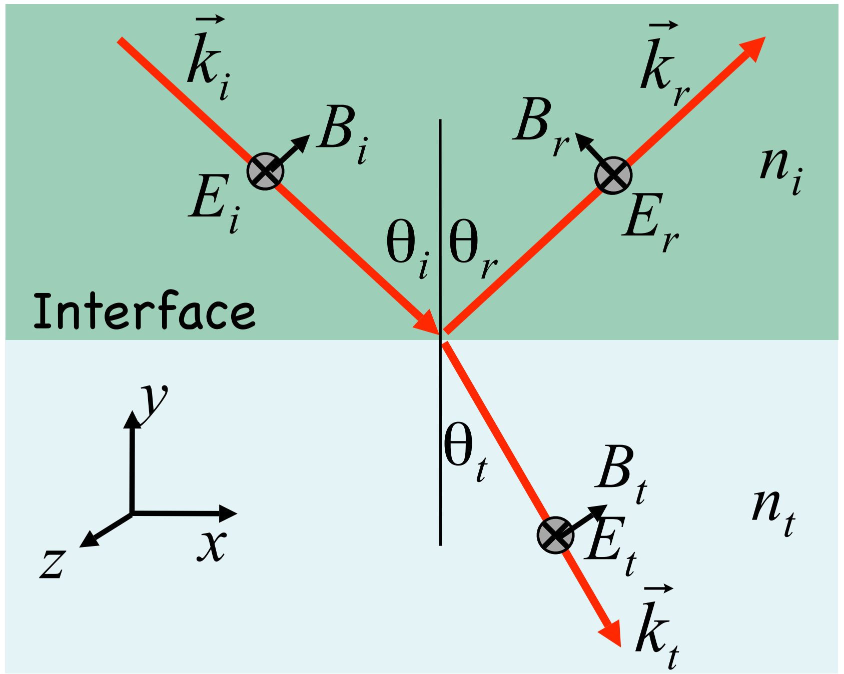
$$n_i\left(E_{0r}-E_{0i}\right)\cos\theta_i=-n_t\left(E_{0r}+E_{0i}\right)\cos\theta_t$$

rearranging to find $r_\perp = E_{or}/E_{oi}$ gives

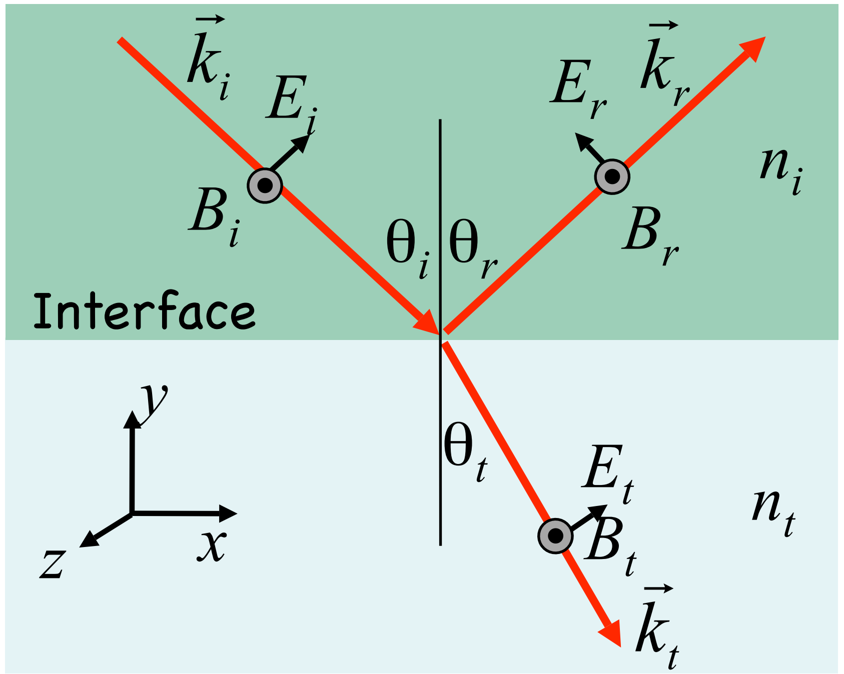
$$r_\perp=\frac{E_{0r}}{E_{0i}}=\frac{n_i\cos\theta_i-n_t\cos\theta_t}{n_i\cos\theta_i+n_t\cos\theta_t}$$

and similarly $t_\perp = E_{ot}/E_{oi}$ is

$$t_\perp=\frac{E_{0t}}{E_{0i}}=\frac{2n_i\cos\theta_i}{n_i\cos\theta_i+n_t\cos\theta_t}$$



P-Polarization at a Boundary



The tangential electric field is continuous

$$E_i(y=0,t)\cos\theta_i + E_r(y=0,t)\cos\theta_r$$
$$= E_t(y=0,t)\cos\theta_t$$

The tangential magnetic field is continuous*

$$B_i(y=0,t) + B_r(y=0,t) = B_t(y=0,t)$$

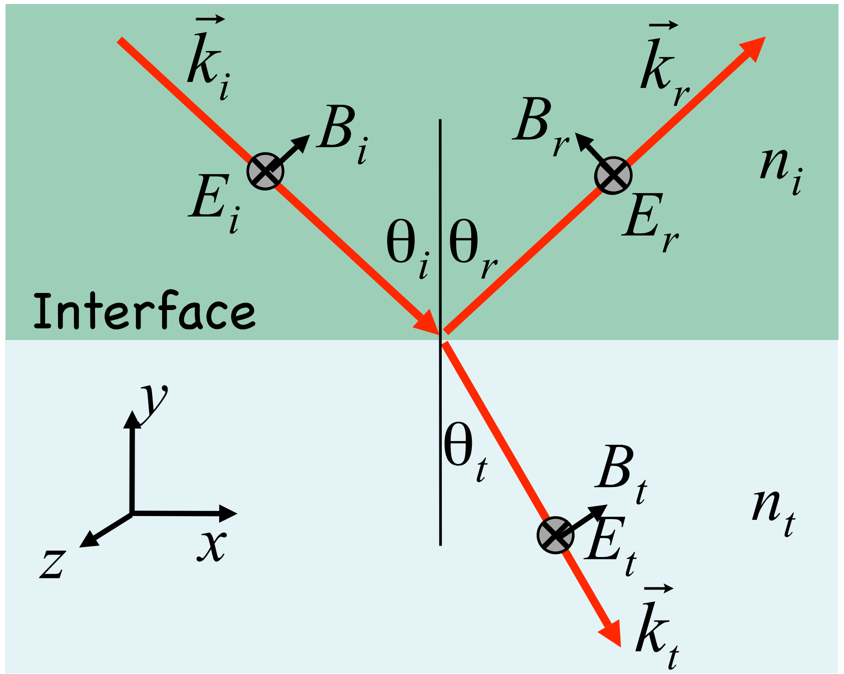
$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$$
$$E_{1\parallel} = E_{2\parallel}$$
$$B_{1\perp} = B_{2\perp}$$
$$\frac{B_{1\parallel}}{\mu_1} = \frac{B_{2\parallel}}{\mu_2}$$

Using $\theta_i=\theta_r$ and $E=cB/n$ and considering only the amplitude of the waves at the boundary

$$n_t\left(E_{0r}-E_{0i}\right)\cos\theta_i = n_i\left(E_{0r}+E_{0i}\right)\cos\theta_t$$

*It's actually the tangential B/μ, but we're assuming $\mu=\mu_0$

P-Polarization at a Boundary



$$n_t\left(E_{0r}-E_{0i}\right)\cos\theta_i = n_i\left(E_{0r}+E_{0i}\right)\cos\theta_t$$

rearranging to find $r_{\parallel} = E_{or}/E_{oi}$ gives

$$r_{\parallel} = \frac{E_{0r}}{E_{0i}} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

and similarly $t_{\parallel} = E_{ot}/E_{oi}$ is

$$t_{\parallel} = \frac{E_{0t}}{E_{0i}} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

Fresnel Equations

- At normal incidence

$$r = \frac{n_t - n_i}{n_t + n_i}$$

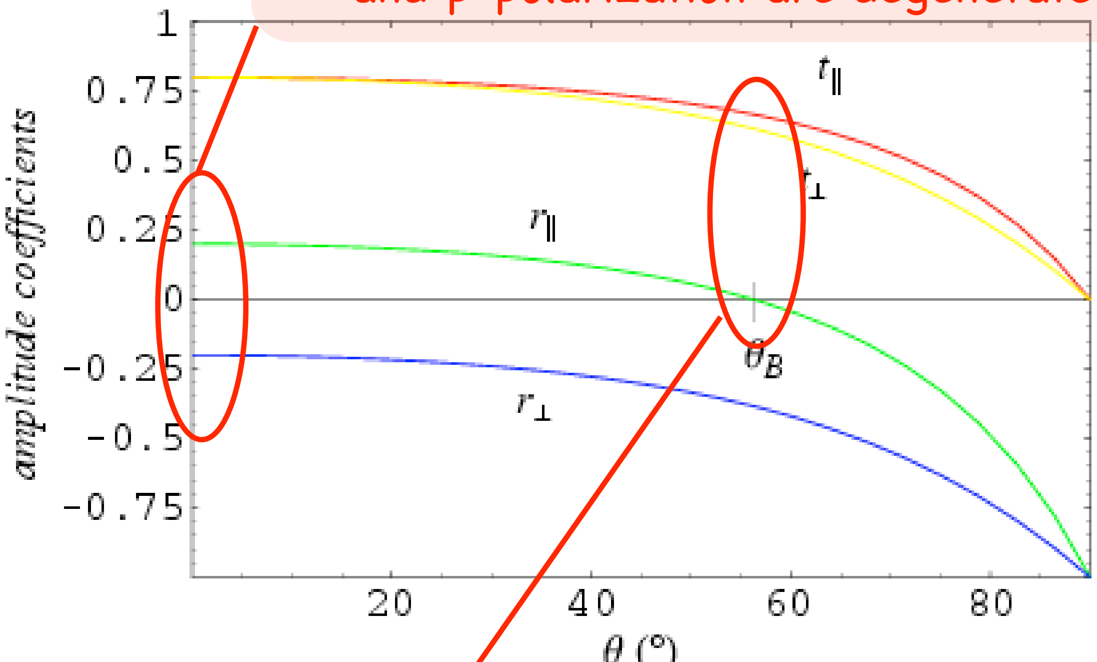
- At "Brewster's angle"

$$r_{\parallel} = 0$$

- At grazing incidence

$$\lim_{\theta_i \to 90^o} r = -1$$

How can $r_{\parallel}$ differ from $r_{\perp}$ at $\theta=0$ where s and p-polarization are degenerate?



reflection and transmission at an air-glass interface

$$r_{\parallel} = \frac{E_{0r}}{E_{0i}} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i}$$

$$t_{\parallel} = \frac{E_{0t}}{E_{0i}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$

$$r_{\perp} = \frac{E_{0r}}{E_{0i}} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_{\perp} = \frac{E_{0t}}{E_{0i}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

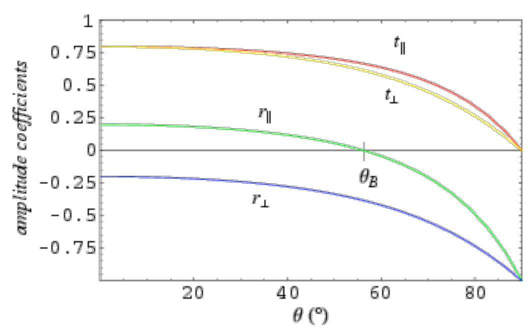
Why isn't $t_{\parallel}=1$ when $r_{\parallel}=1$? If none of the field is reflected, shouldn't it all be transmitted?

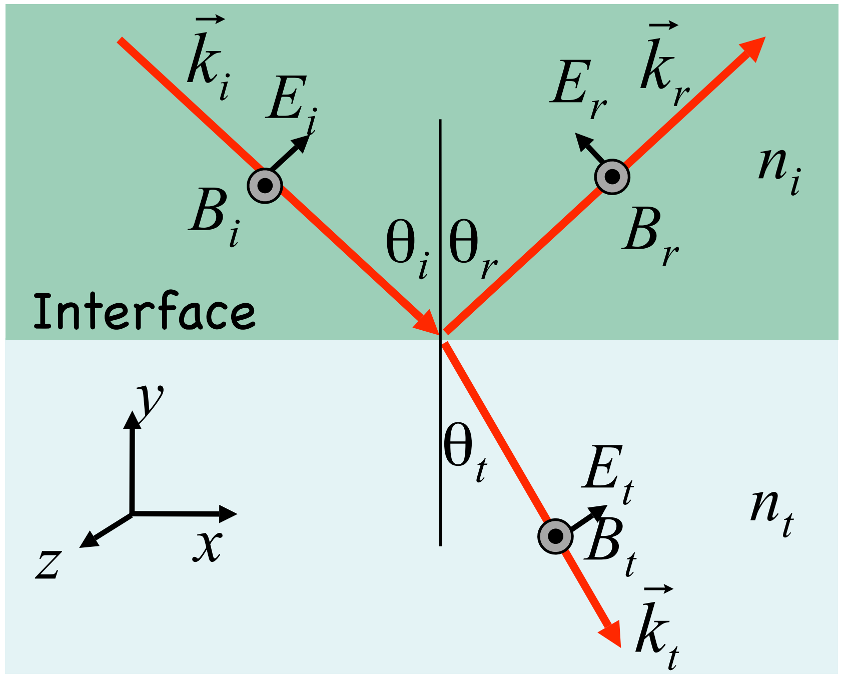
Reflection and Transmission at Normal Incidence

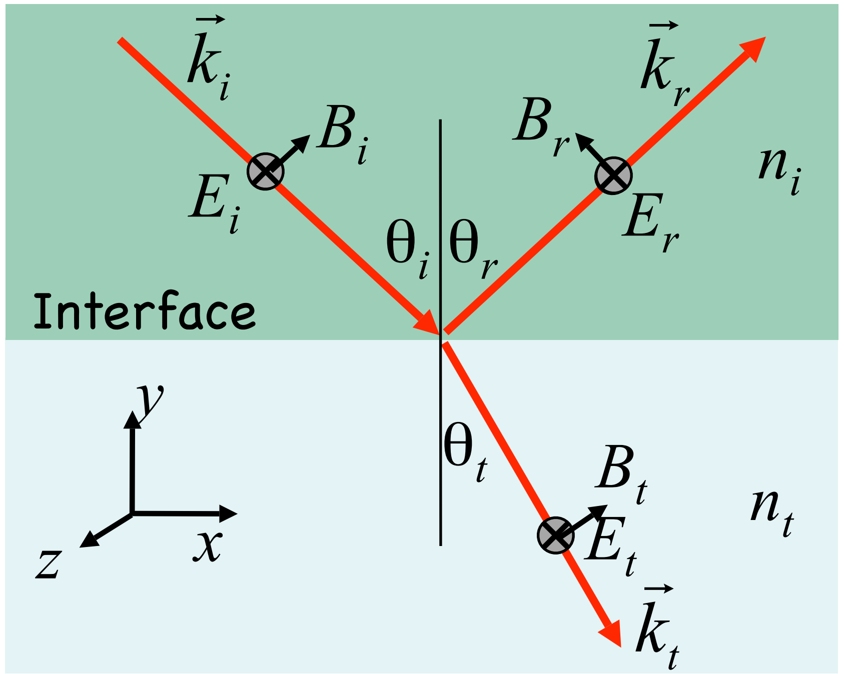
Considering our definition for what we consider positive E_r notice that as $\theta \rightarrow 0$ we have positive values for E_r pointing in different directions for s and p-polarization, hence the reflection coefficients need to have opposite sign for them to converge to the same physical solution

$$r = \frac{n_t - n_i}{n_t + n_i}$$

$$t = \frac{2n_t}{n_t + n_i}$$



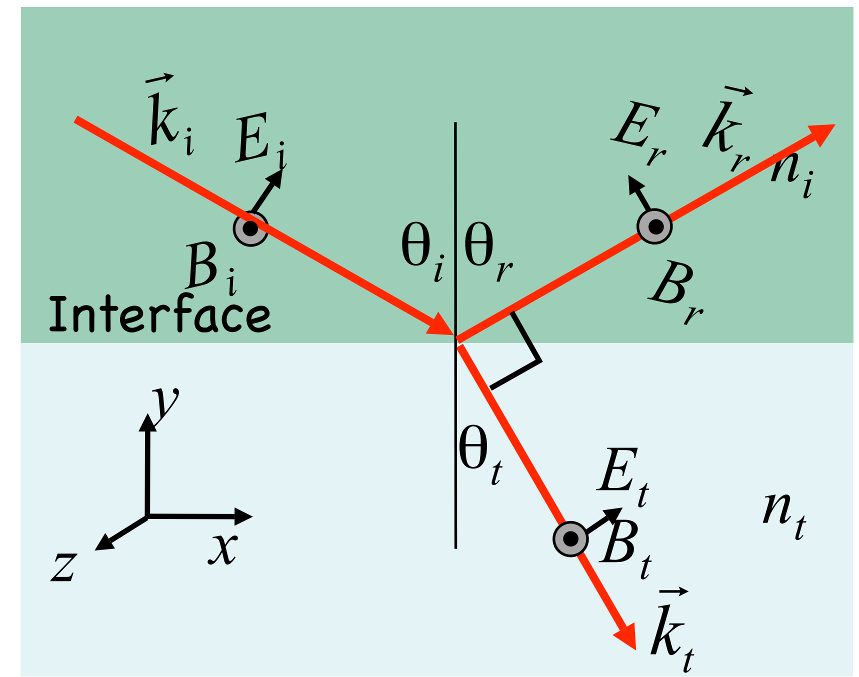




Note that $r^2+t^2=1$ indicating energy is conserved at the boundary

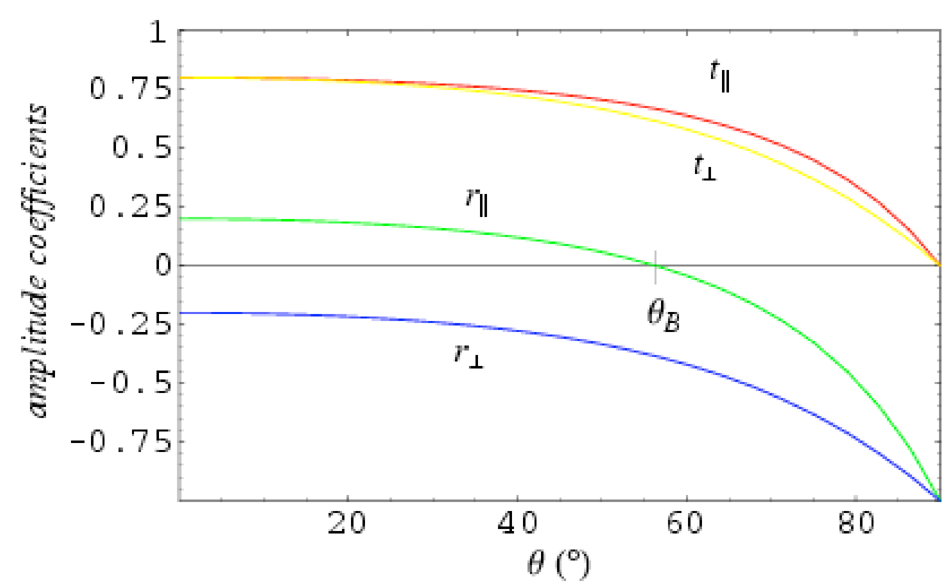
Brewster's Angle

When the incident electric field oscillations excite dipole oscillation in the material in a direction parallel to the reflected beam the dipoles cannot radiate along the direction of the reflected beam

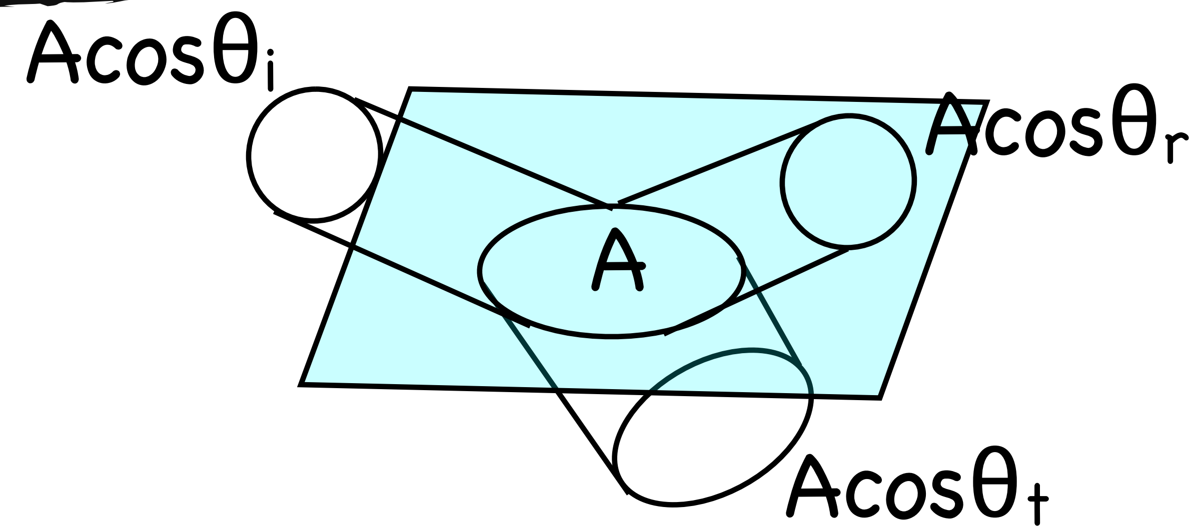


At this angle, called "Brewster's angle" $r_{\parallel}=0$. There are many practical applications of this

- polarize the reflected light
- minimize reflection off the surface of laser mirrors



Conservation of Energy



- Irradiance is proportional to the square of the field so if we are interested in the reflected and transmitted irradiance we use the square of the field **reflectivity r** and **transmissivity t** (i.e. r^2 and t^2)

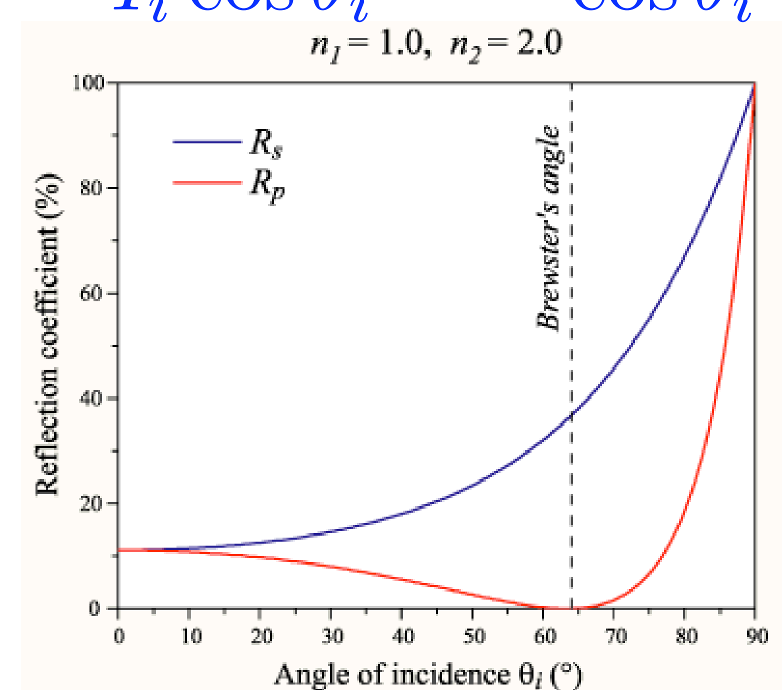
$$R = \frac{I_r \cos\theta_r}{I_i \cos\theta_i} = r^2$$

$$T = \frac{I_t \cos\theta_t}{I_i \cos\theta_i} = t^2 \frac{\cos\theta_t}{\cos\theta_i}$$

- The power is irradiance times area, and the cross sectional area of the beam is different for the incident and transmitted beams

- The power reflection and transmission coefficients for a beam are R and T and are called the **Reflectance** and **Transmittance**

- R+T=1 so energy is conserved



Summary

- A full electromagnetic treatment of the fields at the boundary of two dielectrics leads to the Fresnel equations for transmissivity and reflectivity
- At normal incidence $$r = \frac{n_t - n_i}{n_t + n_i} \quad t = \frac{2n_t}{n_t + n_i}$$
- At Brewster's angle the reflectivity of the P-polarized field goes to zero
- The power reflectivity and transmissivity of a beam are $$R = r^2 \qquad T = t^2 \frac{\cos\theta_t}{\cos\theta_i}$$