# Bloch waves and Bandgaps

Chapter 6

Physics 208, Electro-optics

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# Bloch Waves

There are various classes of boundary conditions for which solutions to the wave equation are not plane waves

- Planar conductor results in standing waves

$$E(z) = 2E_0 \sin(k_z z)\cos(\omega t)$$

- Waveguide and cavities results in modal structure

$$E(x,y,z) = E_{nm}(x,y)e^{-ikz}$$

- Periodic materials result in Bloch waves

$$\vec{E}(\vec{r}) = \int_0^{2\pi/\Lambda} E(\vec{K},\vec{r})e^{-i\vec{K}\cdot\vec{r}}d\vec{K}$$

# Class Outline

- Types of periodic media
- dispersion relation in layered materials
- Bragg reflection
- Coupled mode theory
- Surface waves

# Periodic Media

Many useful materials and devices may have an inhmogenous index of refraction profile that is periodic

- Dielectric stack optical coatings
- Diffraction gratings
- Holograms
- Acousto-optic devices
- Photonic bandgap crystals

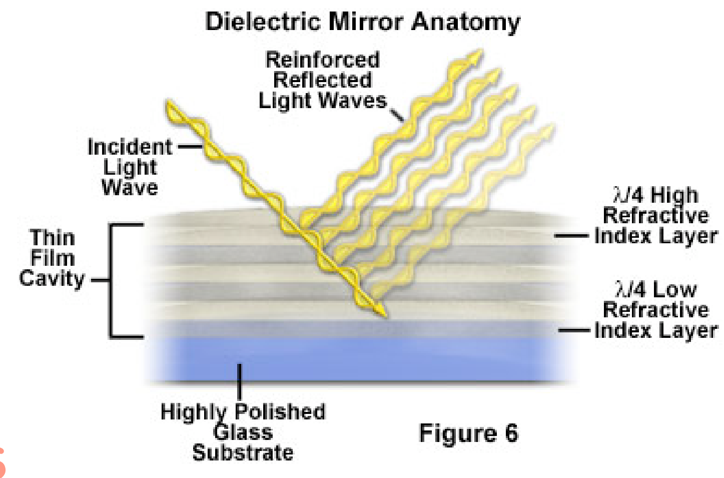

Figure 6

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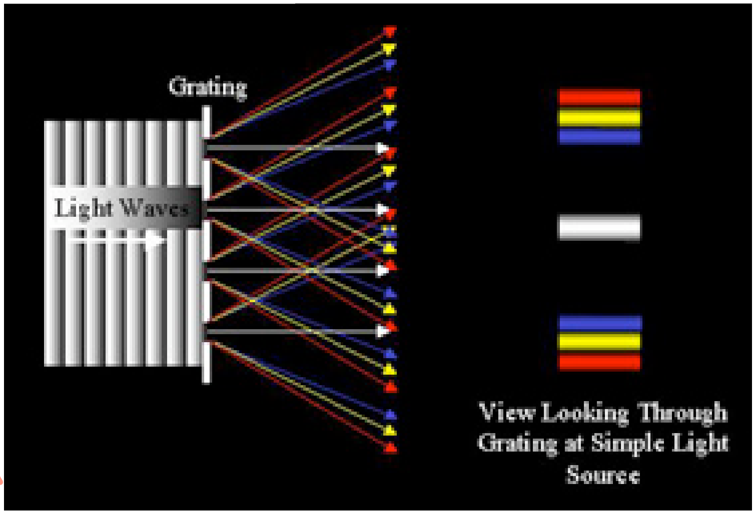

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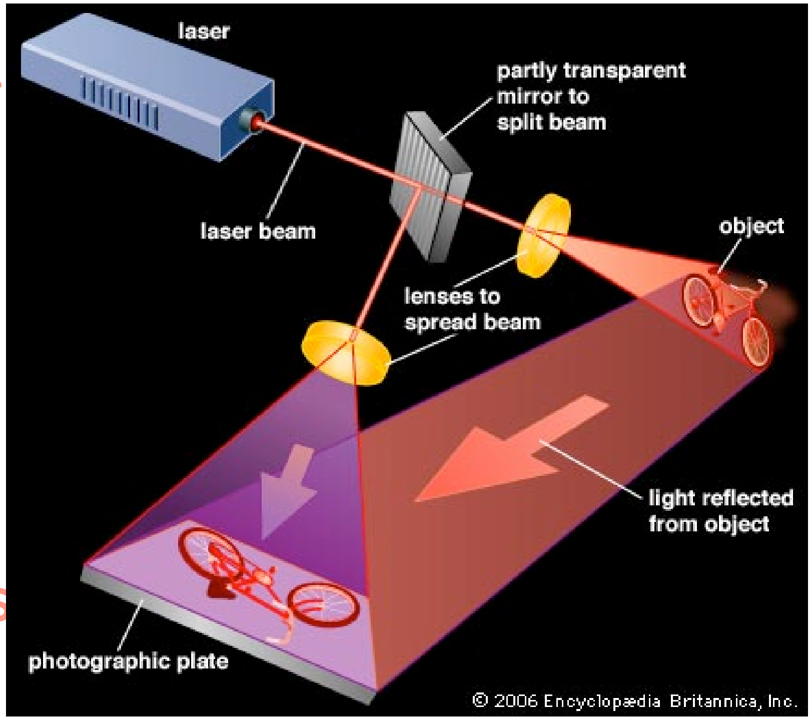

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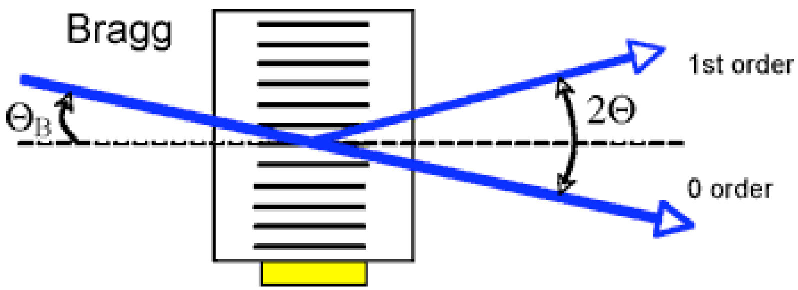

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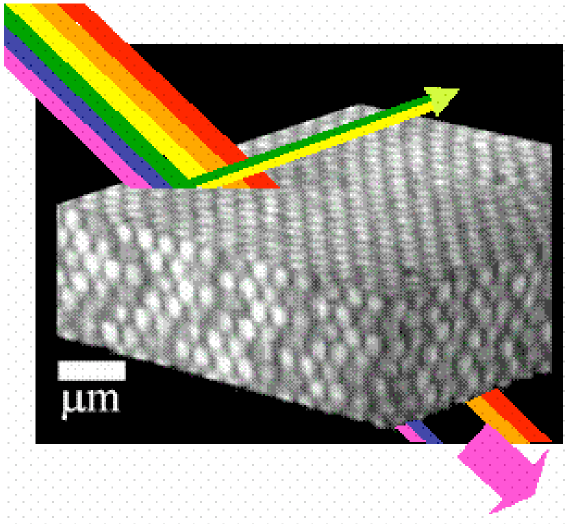
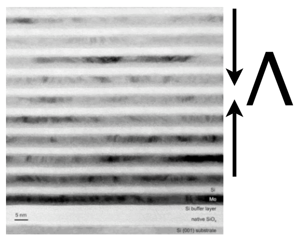


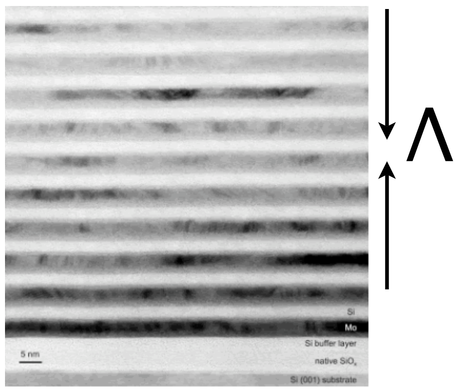
# Bloch's Theorem




$$\vec{E}(\vec{r}) = \int_0^{2\pi/\Lambda} E(\vec{K},\vec{r})e^{-i\vec{K}\cdot\vec{r}}d\vec{K}$$

- Wave solutions in a periodic medium (Bloch waves) are different than in a homogenous medium (plane waves)
- A correction factor E(K,r) accounts for the difference between plane wave solutions and Bloch wave solutions
- Wave amplitude has a periodicity defined by the underlying medium, $E_K(K,r)=E_K(K,r+\Lambda)$
- E(K,r) are normal modes of propagation

# Waves in Layered Media




For a wave normally incident on an isotropic layered material, we'll find the "dispersion relationship" ($\omega$ vs K curve) for Bloch waves. This will tell us about the behavior of waves in the material.

We'll see the periodic structure reflects certain wavelengths. This is referred to as Bragg reflection.

# Wave Equation in Layered Media

Starting with the wave equation

$$\vec{\nabla} \times \left( \vec{\nabla} \times \vec{E} \right) + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

we will plug in the dielectric tensor written as a Fourier series with periodicity $\Lambda$

$$\epsilon(z) = \sum_l \epsilon_l e^{-i\frac{2\pi l}{\Lambda} z}$$

and an arbitrary wave

$$\vec{E} = \int \vec{E}_0(k) e^{-i(kz+\omega t)} dk$$

to get

$$\int k^2 \vec{E}_0(k) e^{-ikz} dk + \omega^2 \mu \sum_l \epsilon_l e^{-i\frac{2\pi l}{\Lambda} z} \int \vec{E}_0(k) e^{-ikz} dk = 0$$

# Wave Equation in Layered Media

with $$\int k^2\vec{E}_0(k)e^{-ikz}dk + \omega^2\mu\sum_l \epsilon_l e^{-i\frac{2\pi l}{\Lambda}z}\int \vec{E}_0(k)e^{-ikz}dk = 0$$

and defining $k' = k + \frac{2\pi l}{\Lambda}$

gives $$\int k^2\vec{E}_0(k)e^{-ikz}dk - \omega^2\mu\int\sum_l \epsilon_l\vec{E}_0(k' - \frac{2\pi l}{\Lambda})e^{-ik'z}dk' = 0$$

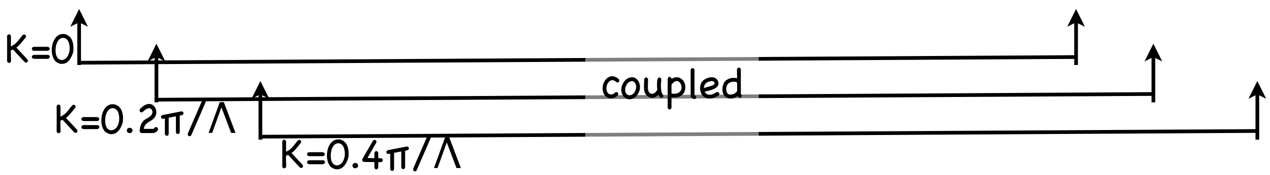
or $$\int\left(k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l\vec{E}_0(k - \frac{2\pi l}{\Lambda})\right)e^{-ikz}dk = 0$$

so $$k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l\vec{E}_0(k - \frac{2\pi l}{\Lambda}) = 0$$ for all k

# Wave Equation in Layered Media

$$k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l \vec{E}_0(k - \frac{2\pi l}{\Lambda}) = 0$$ for all k

Is an infinite set of equations. Consider equations for:
$k\Lambda/2\pi$=0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3 ...




Let K be the value value of $k\pm 2\pi l/\Lambda$ closest to $\omega^2\mu\epsilon_0$ in a series of coupled equations, where $\epsilon_0$ is the zeroth order Fourier coefficient of $\epsilon(z)$.
The whole series of equations for $-\infty<k<\infty$ can be treated instead as a series of coupled equations for $0<K<2\pi/\Lambda$. The solution to each set of equations for a value of K only contains terms at $k=K\pm 2\pi l/\Lambda$, thus

$$\vec{E}(z) = \int \vec{E}_0(k)e^{i(kz+\omega t)}dk \;\rightarrow\; \vec{E}(K,z) = \sum_l \vec{E}_0(K - l\frac{2\pi}{\Lambda})e^{i(K-l\frac{2\pi}{\Lambda})z - i\omega t}$$

# Bloch Waves in Layered Media

The Bloch waves are normal modes of propagation so

$$\vec{E}(z) = \int_0^{2\pi/\Lambda} E(K,z)e^{-iKz}dK$$

and each mode is composed of plane wave components of amplitude $E(K \pm 2\pi l/\Lambda)$. To find these amplitudes we consider the dispersion relation

$$k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l \vec{E}_0(k - \frac{2\pi l}{\Lambda}) = 0$$

Since this represents an infinite set of coupled equations, we will examine this expression and isolate the equations that couple most strongly to $E_0(K)$, ignore the rest and solve for $E_0(k)$

# Field Components in Layered Media

$$k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l \vec{E}_0(k - \frac{2\pi l}{\Lambda}) = 0$$

or $$k^2\vec{E}_0(k) - \omega^2\mu\epsilon_\emptyset\vec{E}_0(k) - \omega^2\mu\epsilon_1\vec{E}_0(k - \frac{2\pi}{\Lambda}) - \omega^2\mu\epsilon_{-1}\vec{E}_0(k + \frac{2\pi}{\Lambda}) - \ldots = 0$$

Allowing us to express the $E_0(K-2\pi l/\Lambda)$ amplitudes as

$$\vec{E}_0(K) = \frac{1}{K^2 - \omega^2\mu\epsilon_\emptyset}\left(\omega^2\mu\epsilon_1\vec{E}_0(K - \frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K + \frac{2\pi}{\Lambda}) - \ldots\right)$$

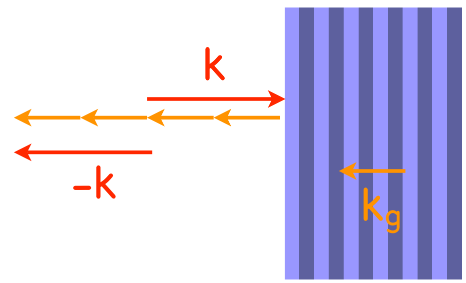
$$\vec{E}_0(K - \frac{2\pi}{\Lambda}) = \frac{1}{(K - \frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_\emptyset}\left(\omega^2\mu\epsilon_1\vec{E}_0(K - 2\frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K) - \ldots\right)$$

$$\vec{E}_0(K + \frac{2\pi}{\Lambda}) = \frac{1}{(K + \frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_\emptyset}\left(\omega^2\mu\epsilon_1\vec{E}_0(K) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K + 2\frac{2\pi}{\Lambda}) - \ldots\right)$$

⋮

# Resonant Coupling of Waves

Momentum of forwards wave with wavenumber k is $\hbar k$, for backwards wave it is $-\hbar k$.




Grating can be thought of as superposition of forwards and backwards going waves, with momenta $\pm\hbar k_g$, where $k_g=2\pi/\Lambda$. For light to couple between forwards and backwards waves, momentum must be conserved $\hbar k+m\hbar k_g=-\hbar k$

This is like a collision of a forward photon with m phonons producing a backwards photon

# Field Components in Layered Media

$$\vec{E}_0(K) = \frac{1}{K^2 - \omega^2\mu\epsilon_{\o}}\left(\omega^2\mu\epsilon_1\vec{E}_0(K - \frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K + \frac{2\pi}{\Lambda}) - \dots\right)$$

$$\vec{E}_0(K - \frac{2\pi}{\Lambda}) = \frac{1}{(K - \frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_{\o}}\left(\omega^2\mu\epsilon_1\vec{E}_0(K - 2\frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K) - \dots\right)$$

$$\vec{E}_0(K + \frac{2\pi}{\Lambda}) = \frac{1}{(K + \frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_{\o}}\left(\omega^2\mu\epsilon_1\vec{E}_0(K) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K + 2\frac{2\pi}{\Lambda}) - \dots\right)$$

In the case where $\left(K - l\frac{2\pi}{\Lambda}\right)^2 \neq \omega^2\mu\epsilon_{\o}$ i.e. the forward wave momentum cannot be converted to the backward wave momentum by the addition of a kick from the grating in the layered material, only the l=0 term is significant and the dispersion relation $K^2\vec{E}_0(K) - \omega^2\mu\sum_l \epsilon_l\vec{E}_0(K - \frac{2\pi l}{\Lambda}) = 0$ for any value of K is uncoupled to that for other values of K, and gives $K^2 - \omega^2\mu\epsilon_{\o} = 0$ meaning the phase velocity is that due to the average index of refraction for the medium

# Coupling of Field Components

$$\vec{E}_0(K+\frac{2\pi l}{\Lambda}) = \frac{1}{(K+\frac{2\pi l}{\Lambda})^2 - \omega^2\mu\epsilon_\o}\left(\omega^2\mu\epsilon_1\vec{E}_0(K+\frac{2\pi(l-1)}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K+2\frac{2\pi l}{\Lambda}) - \ldots\right)$$

l=0 $$\vec{E}_0(K) = \frac{1}{K^2 - \omega^2\mu\epsilon_\o}\left(\omega^2\mu\epsilon_1\vec{E}_0(K-\frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K+\frac{2\pi}{\Lambda}) - \ldots\right)$$

l=-1 $$\vec{E}_0(K-\frac{2\pi}{\Lambda}) = \frac{1}{(K-\frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_\o}\left(\omega^2\mu\epsilon_1\vec{E}_0(K-2\frac{2\pi}{\Lambda}) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K) - \ldots\right)$$

l=+1 $$\vec{E}_0(K+\frac{2\pi}{\Lambda}) = \frac{1}{(K+\frac{2\pi}{\Lambda})^2 - \omega^2\mu\epsilon_\o}\left(\omega^2\mu\epsilon_1\vec{E}_0(K) + \omega^2\mu\epsilon_{-1}\vec{E}_0(K+2\frac{2\pi}{\Lambda}) - \ldots\right)$$

In the case where $\left(K - l\frac{2\pi}{\Lambda}\right)^2 \approx \omega^2\mu\epsilon_\o$ for some non-zero value of l=m this l=m term is also significant and for the dispersion relation

$$K^2\vec{E}_0(K) - \omega^2\mu\sum_l \epsilon_l\vec{E}_0(K-\frac{2\pi l}{\Lambda}) = 0$$

we need only consider two values of K, i.e. K and K-2πm/Λ.

# Dispersion Relation in Layered Media

The dispersion relation $k^2\vec{E}_0(k) - \omega^2\mu\sum_l \epsilon_l \vec{E}_0(k - \frac{2\pi l}{\Lambda}) = 0$

Considering only terms with E(K) and E(K–2πm/Λ) gives

$$\begin{aligned}\left(K^2 - \omega^2\mu\epsilon_\varnothing\right)\vec{E}_0(K) - \omega^2\mu\epsilon_m\vec{E}_0(K - \frac{2\pi m}{\Lambda}) &= 0\\ \left[\left(K - \frac{2\pi m}{\Lambda}\right)^2 - \omega^2\mu\epsilon_\varnothing\right]\vec{E}_0\left(K - \frac{2\pi m}{\Lambda}\right) - \omega^2\mu\epsilon_{-m}\vec{E}_0(K) &= 0\end{aligned}$$

A nontrivial solution to these coupled equations only exists if

$$\begin{vmatrix} K^2 - \omega^2\mu\epsilon_\varnothing & -\omega^2\mu\epsilon_m \\ -\omega^2\mu\epsilon_{-m} & (K - \frac{2\pi m}{\Lambda})^2 - \omega^2\mu\epsilon_\varnothing \end{vmatrix} = 0$$

and for $\epsilon_{-m}=\epsilon_m^*$ in a lossless medium, since $\epsilon_m = \frac{1}{\Lambda}\int_0^\Lambda \epsilon(z)e^{-i2\pi mz/\Lambda}dz$

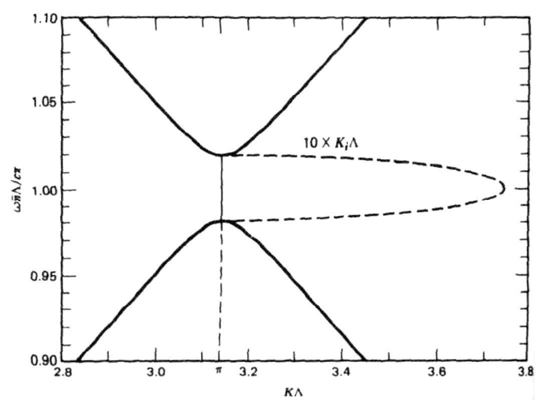
$$(K^2 - \omega^2\mu\epsilon_\varnothing)\left(\left(K - \frac{2\pi m}{\Lambda}\right)^2 - \omega^2\mu\epsilon_\varnothing\right) - \left(\omega^2\mu|\epsilon_m|\right)^2 = 0$$

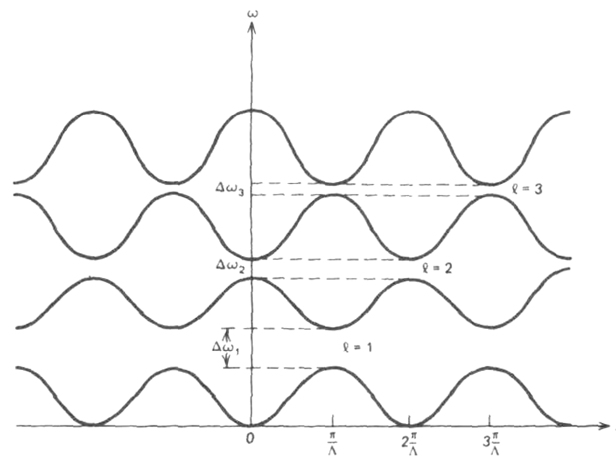
# Dispersion Relation in Layered Media

$$(K^2 - \omega^2\mu\epsilon_\varnothing)\left(\left(K - \frac{2\pi m}{\Lambda}\right)^2 - \omega^2\mu\epsilon_\varnothing\right) - \left(\omega^2\mu|\epsilon_m|\right)^2 = 0$$

which can be solved for K, the Bloch wave vector for a wave of frequency ω

$$K = \frac{m\pi}{\Lambda} \pm \sqrt{\left(\frac{m\pi}{\Lambda}\right)^2 + \epsilon_\varnothing\mu\omega^2 \pm \sqrt{(|\epsilon_m|\mu\omega^2)^2 + \left(\frac{2\pi m}{\Lambda}\right)^2 \epsilon_\varnothing\mu\omega^2}}$$




graph of dispersion relationship (figure 6.2)

# Bandgaps in Layered Media

$$K = \frac{m\pi}{\Lambda} \pm \sqrt{\left(\frac{m\pi}{\Lambda}\right)^2 + \epsilon_\o \mu\omega^2 \pm \sqrt{\left(|\epsilon_m|\mu\omega^2\right)^2 + \left(\frac{2\pi m}{\Lambda}\right)^2 \epsilon_\o \mu\omega^2}}$$

When the Bragg condition is met ($K-2\pi m/\Lambda=\omega^2\epsilon_\o\,\mu$)

real solutions exist for $\omega^2 < \frac{K^2}{\mu\left(\epsilon_\o + |\epsilon_m|\right)}$ and $\omega^2 > \frac{K^2}{\mu\left(\epsilon_\o - |\epsilon_m|\right)}$

Solutions for $\frac{K^2}{\mu\left(\epsilon_\o + |\epsilon_m|\right)} < \omega^2 < \frac{K^2}{\mu\left(\epsilon_\o - |\epsilon_m|\right)}$

are complex, this region is called the forbidden band. At the center of the forbidden band where $\left(K - m\frac{2\pi}{\Lambda}\right)^2 \approx \omega^2\mu\epsilon_\o$ and $K^2 - \omega^2\mu\epsilon_\o = 0$ , i.e. $\omega^2 = \frac{(m\pi)^2}{\Lambda^2\mu\epsilon_\o}$

the dispersion relation gives $K = \frac{m\pi}{\Lambda}\left(1 \pm i\frac{|\epsilon_1|}{2\epsilon_\o}\right)$

# Bandgap Properties

The forbidden band has a width in ω, called the bandgap that is

$$\Delta\omega_{gap} = \omega\frac{|\epsilon_m|}{\epsilon_\emptyset}$$

And at its center has an attenuation coefficient

$$\mathrm{Im}[k] = \frac{m\pi}{2\Lambda}\frac{\Delta\omega_{gap}}{\omega}$$

Thus, the greater the Fourier coefficient $|\epsilon_m|$ the larger the bandgap and the stronger the attenuation in the gap.

# Bloch Waveform

With our calculated dispersion relationship

$$K = \frac{m\pi}{\Lambda} \pm \sqrt{\left(\frac{m\pi}{\Lambda}\right)^2 + \epsilon_{\o}\mu\omega^2 \pm \sqrt{(|\epsilon_m|\mu\omega)^2 + \left(\frac{2\pi m}{\Lambda}\right)^2 \epsilon_{\o}\mu\omega^2}}$$

we can choose a frequency ω, calculate K(ω) and solve

$$\begin{aligned} \left(K^2 - \omega^2\mu\epsilon_{\o}\right)\vec{E}_0(K) - \omega^2\mu\epsilon_m\vec{E}_0(K - \frac{2\pi m}{\Lambda}) &= 0 \\ \left[\left(K - \frac{2\pi m}{\Lambda}\right)^2 - \omega^2\mu\epsilon_{\o}\right]\vec{E}_0\left(K - \frac{2\pi m}{\Lambda}\right) - \omega^2\mu\epsilon_{-m}\vec{E}_0(K) &= 0 \end{aligned}$$

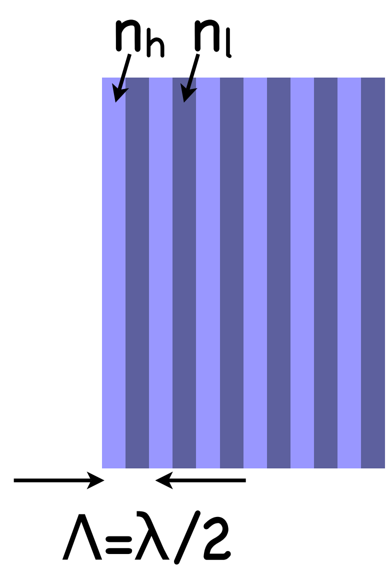
for E(K) and E(K-2πm/Λ) giving the waveform of the Bloch mode at frequency ω (or wavenumber K(ω)) considering only these two components

$$\vec{E}(K,z) \approx \sum_{l=0,m} \vec{E}_0(K - l\frac{2\pi}{\Lambda})e^{-i\left(K - l\frac{2\pi}{\Lambda}\right)z - i\omega t}$$

# Bloch Waveform Example

Consider a periodic structure consisting of alternating layers of high index and low index material ($n_h$=1.8, $n_l$=1.5).

Find the waveform in the material for a wave of wavelength $\lambda=2\Lambda$



# Bloch Waveform Example

Example solved using "Mathematica"

```
(*Dispersion relation for |k- 2 Pi m/Λ|~k with only two terms l=0 and l=m*)
eqd = (k^2 - ω^2 μ ep0) ((k - 2 Pi m / Λ)^2 - ω^2 μ ep0) - (ω μ epm)^2 == 0
```

Out[717]= $-\text{epm}^2\,\mu^2\,\omega^2 + \left(k^2 - \text{ep0}\,\mu\,\omega^2\right)\left(\left(k - \frac{2\,m\,\pi}{\Lambda}\right)^2 - \text{ep0}\,\mu\,\omega^2\right) == 0$

```
(*Dispersion relation expressed as k (w) and w (k)*)
sol = Solve[eqd, k]
solw = Solve[eqd, ω]
```

Out[718]=
$$\left\{\left\{k \to \frac{m\,\pi - \sqrt{m^2\,\pi^2 + \text{ep0}\,\Lambda^2\,\mu\,\omega^2 - \Lambda\sqrt{4\,\text{ep0}\,m^2\,\pi^2\,\mu\,\omega^2 + \text{epm}^2\,\Lambda^2\,\mu^2\,\omega^2}}}{\Lambda}\right\}, \left\{k \to \frac{m\,\pi + \sqrt{m^2\,\pi^2 + \text{ep0}\,\Lambda^2\,\mu\,\omega^2 - \Lambda\sqrt{4\,\text{ep0}\,m^2\,\pi^2\,\mu\,\omega^2 + \text{epm}^2\,\Lambda^2\,\mu^2\,\omega^2}}}{\Lambda}\right\},\right.$$
$$\left.\left\{k \to \frac{m\,\pi - \sqrt{m^2\,\pi^2 + \text{ep0}\,\Lambda^2\,\mu\,\omega^2 + \Lambda\sqrt{4\,\text{ep0}\,m^2\,\pi^2\,\mu\,\omega^2 + \text{epm}^2\,\Lambda^2\,\mu^2\,\omega^2}}}{\Lambda}\right\}, \left\{k \to \frac{m\,\pi + \sqrt{m^2\,\pi^2 + \text{ep0}\,\Lambda^2\,\mu\,\omega^2 + \Lambda\sqrt{4\,\text{ep0}\,m^2\,\pi^2\,\mu\,\omega^2 + \text{epm}^2\,\Lambda^2\,\mu^2\,\omega^2}}}{\Lambda}\right\}\right\}$$

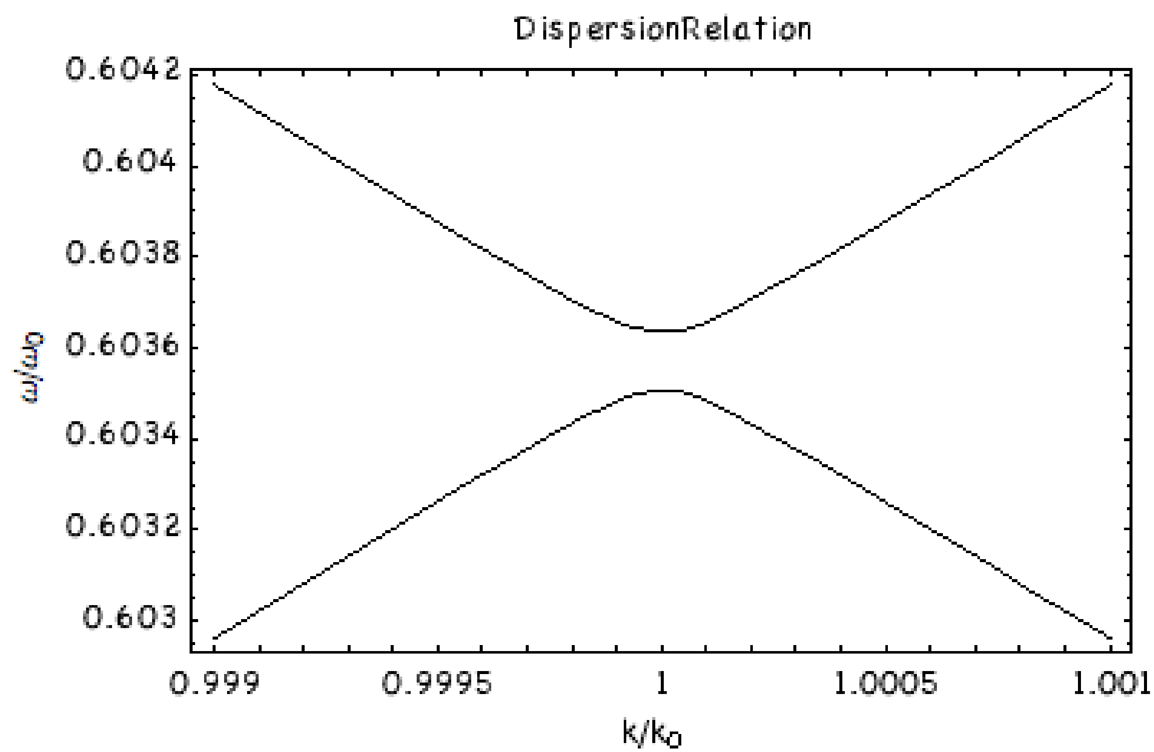
Out[719]=
$$\left\{\left\{\omega \to -\frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{\text{epm}^2}{\text{ep0}^2} + \frac{2\,k^2}{\text{ep0}\,\mu} + \frac{4\,m^2\,\pi^2}{\text{ep0}\,\Lambda^2\,\mu} - \frac{4\,k\,m\,\pi}{\text{ep0}\,\Lambda\,\mu} - \frac{1}{\text{ep0}^2\,\Lambda^2\,\mu^2}\right.}\right.\right.\right.$$
$$\left.\left.\left.\left(\sqrt{\left(-4\,\text{ep0}^2\,\Lambda^2\,\left(4\,k^2\,m^2\,\pi^2 - 4\,k^3\,m\,\pi\,\Lambda + k^4\,\Lambda^2\right)\mu^2 + \left(-4\,\text{ep0}\,m^2\,\pi^2\,\mu + 4\,\text{ep0}\,k\,m\,\pi\,\Lambda\,\mu - 2\,\text{ep0}\,k^2\,\Lambda^2\,\mu - \text{epm}^2\,\Lambda^2\,\mu^2\right)^2\right)}\right)\right)}\right)\right\}, \left\{\omega \to \frac{1}{\sqrt{2}}\left(\sqrt{\left(\frac{\text{epm}^2}{\text{ep0}^2} + \frac{2\,k^2}{\text{ep0}\,\mu} +\right.}\right.\right.$$
$$\left.\left.\left.\frac{4\,m^2\,\pi^2}{\text{ep0}\,\Lambda^2\,\mu} - \frac{4\,k\,m\,\pi}{\text{ep0}\,\Lambda\,\mu} - \frac{1}{\text{ep0}^2\,\Lambda^2\,\mu^2}\left(\sqrt{\left(-4\,\text{ep0}^2\,\Lambda^2\,\left(4\,k^2\,m^2\,\pi^2 - 4\,k^3\,m\,\pi\,\Lambda + k^4\,\Lambda^2\right)\mu^2 + \left(-4\,\text{ep0}\,m^2\,\pi^2\,\mu + 4\,\text{ep0}\,k\,m\,\pi\,\Lambda\,\mu - 2\,\text{ep0}\,k^2\,\Lambda^2\,\mu - \text{epm}^2\,\Lambda^2\,\mu^2\right)^2\right)}\right)\right)}\right)\right\},$$
$$\left\{\omega \to -\sqrt{\left(\frac{\text{epm}^2}{2\,\text{ep0}^2} + \frac{k^2}{\text{ep0}\,\mu} + \frac{2\,m^2\,\pi^2}{\text{ep0}\,\Lambda^2\,\mu} - \frac{2\,k\,m\,\pi}{\text{ep0}\,\Lambda\,\mu} + \frac{1}{2\,\text{ep0}^2\,\Lambda^2\,\mu^2}\left(\sqrt{\left(-4\,\text{ep0}^2\,\Lambda^2\,\left(4\,k^2\,m^2\,\pi^2 - 4\,k^3\,m\,\pi\,\Lambda + k^4\,\Lambda^2\right)\mu^2 +\right.}\right.\right.}\right.$$
$$\left.\left.\left.\left.\left(-4\,\text{ep0}\,m^2\,\pi^2\,\mu + 4\,\text{ep0}\,k\,m\,\pi\,\Lambda\,\mu - 2\,\text{ep0}\,k^2\,\Lambda^2\,\mu - \text{epm}^2\,\Lambda^2\,\mu^2\right)^2\right)\right)\right)}\right\}, \left\{\omega \to \sqrt{\left(\frac{\text{epm}^2}{2\,\text{ep0}^2} + \frac{k^2}{\text{ep0}\,\mu} + \frac{2\,m^2\,\pi^2}{\text{ep0}\,\Lambda^2\,\mu} - \frac{2\,k\,m\,\pi}{\text{ep0}\,\Lambda\,\mu} +\right.}\right.$$
$$\left.\left.\left.\frac{1}{2\,\text{ep0}^2\,\Lambda^2\,\mu^2}\left(\sqrt{\left(-4\,\text{ep0}^2\,\Lambda^2\,\left(4\,k^2\,m^2\,\pi^2 - 4\,k^3\,m\,\pi\,\Lambda + k^4\,\Lambda^2\right)\mu^2 + \left(-4\,\text{ep0}\,m^2\,\pi^2\,\mu + 4\,\text{ep0}\,k\,m\,\pi\,\Lambda\,\mu - 2\,\text{ep0}\,k^2\,\Lambda^2\,\mu - \text{epm}^2\,\Lambda^2\,\mu^2\right)^2\right)}\right)\right)}\right\}\right\}$$

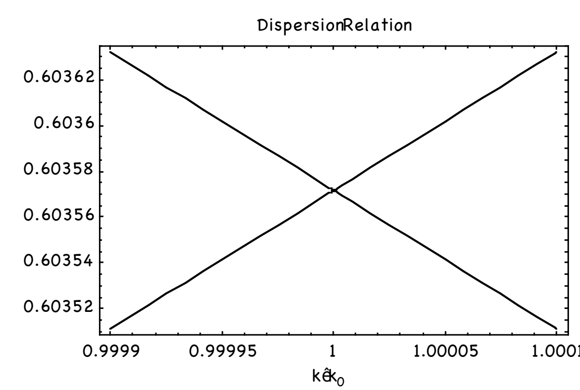
# Bloch  Waveform Example

```
(* assign each of the 4 possible solutions to the dispersion relationship to different variables k1-k4 for k (w),
 w1-w4 for w (k)*)
k1 = k /. sol[[1]][[1]];
k2 = k /. sol[[2]][[1]];
k3 = k /. sol[[3]][[1]];
k4 = k /. sol[[4]][[1]];
w1 = ω /. solw[[1]][[1]];
w2 = ω /. solw[[2]][[1]];
w3 = ω /. solw[[3]][[1]];
w4 = ω /. solw[[4]][[1]];

(*parameters to use in numerical solutions*)
nl = 1.5; (*Low index layer*)
nh = 1.8; (*High index layer*)
epl = nl^2;
eph = nh^2;
λ = 1064 10^-9; (*wavelength*)
c = 299792458;
ω0 = 2 Pi c / λ;
k0 = 2 Pi / λ;

(*List of assignment of numerical values to parameters*)
var = {ep0 → 8.854187817 10^-12 * (epl + eph) / 2, μ → 4 Pi 10^-7, Λ → 0.5 λ, m → 1, epm → 8.854187817 10^-12 * 2 / Pi (eph - epl),
  k → k0 * x}
(*This list Exagerates the size of εm by 10^12 so that structure of bandgap can be seen and is not  lost
  due to numerical precision limits*)
varfake = {ep0 → 8.854187817 10^-12 * (epl + eph) / 2, μ → 4 Pi 10^-7, Λ → 0.5 λ, m → 1,
  epm → 10^12 8.854187817 10^-12 * 2 / Pi (eph - epl), k → k0 * x}
```

# Bloch Waveform Example

```
(*Plot of dispersion relationship with bandgap exagerated by 10^12 (see note about varfake above)*)
dx = 0.001; (*width of graph*)
Pl1 = Plot[{ w2 / ω0 /. varfake, w4 / ω0 /. varfake}, {x, 1 - dx, 1 + dx}, DisplayFunction → Identity]
Plot[{ w2 / ω0 /. varfake, w4 / ω0 /. varfake}, {x, 1 - dx, 1 + dx}, PlotLabel → "Dispersion Relation",
 FrameLabel → {"k/k0", "ω/ω0"}, TextStyle → {FontFamily → "Chalkboard", FontSize → 12}, ImageSize → {400, 300 },
 Frame → True, FrameTicks → Automatic]
```



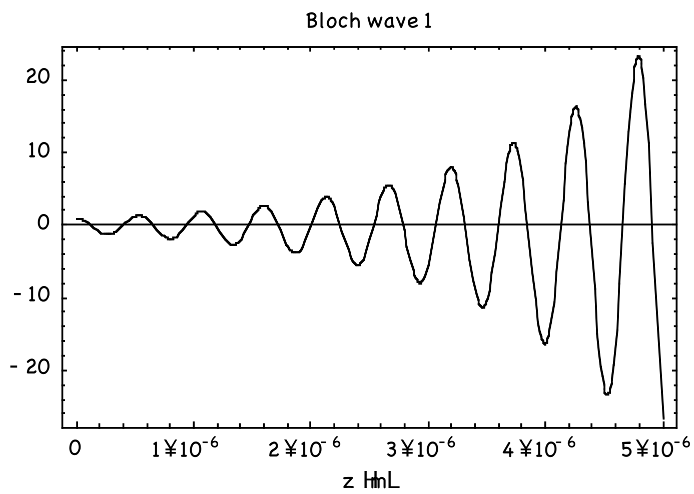

Dispersion relation without exaggeration

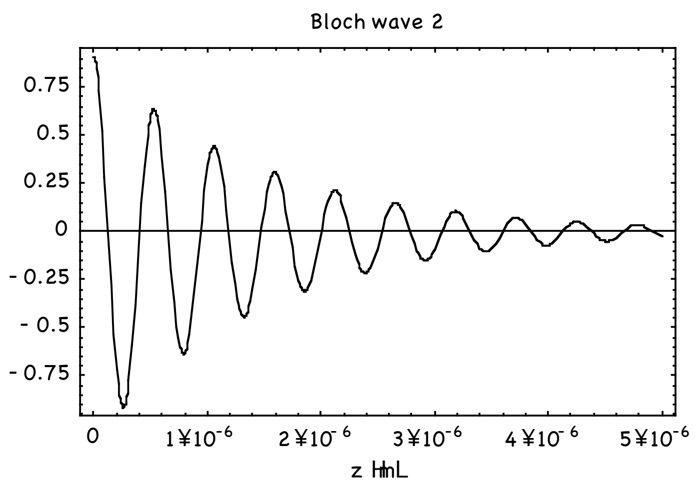
# Bloch Waveform Example

```
In[929]:= (*Bloch wavenumber and frequency at the center of the bandgap*)

In[930]:= K1 = Pi / Λ (1 + I Abs[epm] / (2 ep0)) /. var
          K2 = Pi / Λ (1 - I Abs[epm] / (2 ep0)) /. var
          wc = 2 c Pi / ((nl + nh) Λ ) /. var;

Out[930]= 5.90525 × 10^6 + 677924. i

Out[931]= 5.90525 × 10^6 - 677924. i

In[938]:= (*Equations relating amplitude of E (k) and E (k- 2 Pi m/Λ) components of Bloch wave*)
          eq1 = (K^2 - ω^2 μ ep0) Eo - ω^2 μ epm Em == 0;
          eq2 = ((K - 2 Pi m / Λ)^2 - ω^2 μ ep0) Em - ω^2 μ epm Eo == 0;

In[943]:= (*E0 and Em are the plane wave component amplitudes of the Bloch wave in the periodic mateiral.  There
            are two solutions for Em realtive to E0, one for each value of K*)
          E0 = 1;
          Em1 = Em /. Solve[{eq1 /. Join[var, {ω → wc, K → K1}], Eo == 1}, Em][[1]][[1]]
          Em2 = Em /. Solve[{eq1 /. Join[var, {ω → wc, K → K2}], Eo == 1}, Em][[1]][[1]]

Out[944]= -0.0926296 + 0.991803 i

Out[945]= -0.0926296 - 0.991803 i
```

# Bloch Waveform Example

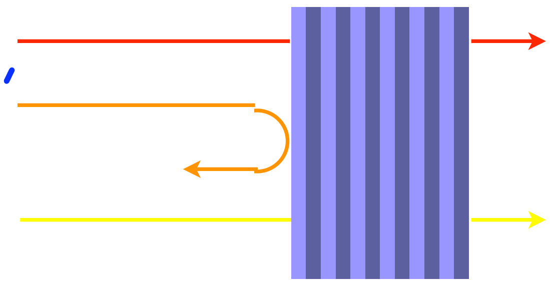
```
In[970]:= (*Bloch waveform in the periodic material for both possible wavenumbers at the bandgap center (K1 and K2)*)
E1 = (E0 Exp[-I k0 z] + Em1 Exp[-I (k0 - 2 Pi m / Λ) z]) Exp[-I K1 z] /. var;
E2 = (E0 Exp[-I k0 z] + Em2 Exp[-I (k0 - 2 Pi m / Λ) z]) Exp[-I K2 z] /. var;
Plot[Re[E1], {z, 0, 5 10^-6}, PlotLabel → "Bloch wave 1", FrameLabel → {"z (m)", "relative amplitude"},
 TextStyle → {FontFamily → "Chalkboard", FontSize → 12}, ImageSize → {400, 300 }, Frame → True, FrameTicks → Automatic]
Plot[Re[E2], {z, 0, 5 10^-6}, PlotLabel → "Bloch wave 2", FrameLabel → {"z (m)", "relative amplitude"},
 TextStyle → {FontFamily → "Chalkboard", FontSize → 12}, ImageSize → {400, 300 }, Frame → True, FrameTicks → Automatic]
```



# High-Reflector Stack

A series of alternating high-index, low-index layers, each λ/4 in thickness has



$$\left(k - \frac{2\pi}{\Lambda}\right)^2 = \omega^2 \mu \epsilon_{\emptyset}$$ and $\epsilon_1 \neq 0$, and $$\omega^2 = \frac{\pi^2}{\Lambda^2 \mu \epsilon_{\emptyset}}$$

therefor light with wavenumber $k=2\pi/\lambda$ cannot propagate through the medium. Instead it is resonantly coupled to a wave with wavenumber $k=-2\pi/\lambda$, i.e. a backwards traveling wave: The medium acts as a reflector for specific wavelengths, this is the principle behind high-reflectivity dielectric coatings.

# Alternative Methods

In our previous method we solved the 1D wave equation for all frequencies of plane waves to get the dispersion relation.

Alternatively we can consider wave propagation in the material, impose boundary conditions at the interfaces and require self-consistent solutions to get the dispersion relation - We will apply this method for a solution in 2D to the dielectric stack problem

# Boundary Conditions

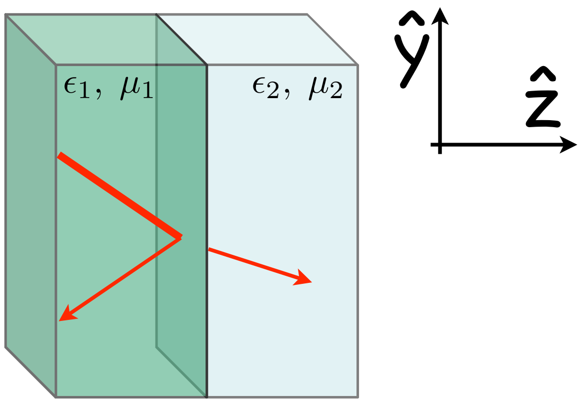
When an EM wave propagates across an interface, Maxwell's equations must be satisfied at the interface as well as in the bulk materials. The constraints necessary for this to occur are called the "boundary conditions"

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

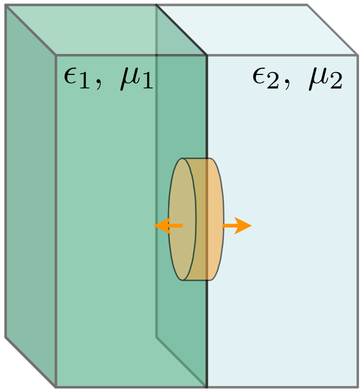
$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$



# Boundary Conditions

Gauss' law can be used to find the boundary conditions on the component of the electric field that is perpendicular to the interface.

If the materials are dielectrics there will be no free charge on the surface (q=0)




$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

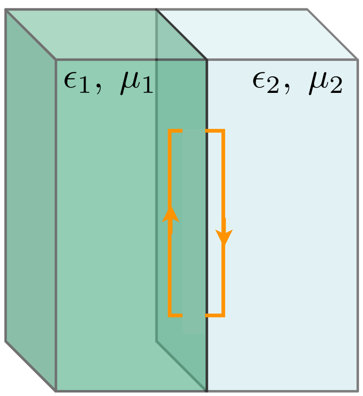
$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

$$\epsilon_1 E_{1z} A - \epsilon_2 E_{2z} A = \sum q^{\,0} \quad \therefore \quad \epsilon_1 E_{1z} = \epsilon_2 E_{2z}$$

# Boundary Conditions

Faraday's law can be applied at the interface. If the loop around which the electric field is computed is made to have an infintesimal area the right side will go to zero giving a relationship between the parallel components of the electric field




$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

$$E_{2x,y}L - E_{1x,y}L = -\frac{d}{dt}\int \cancelto{0}{B} \cdot dA \quad \therefore \quad E_{1x,y} = E_{2x,y}$$

# Boundary Conditions

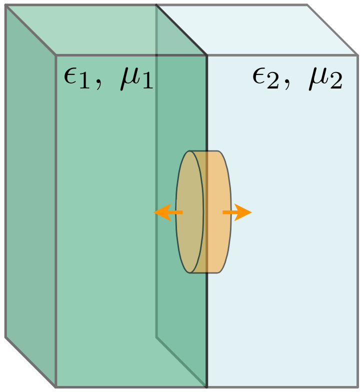
Gauss' law for magnetism gives a relationship between the perpendicular components of the magnetic field at the interface

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$




$$B_{1z}A - B_{2z}A = 0 \qquad \therefore \qquad B_{1z} = B_{2z}$$

# Boundary Conditions

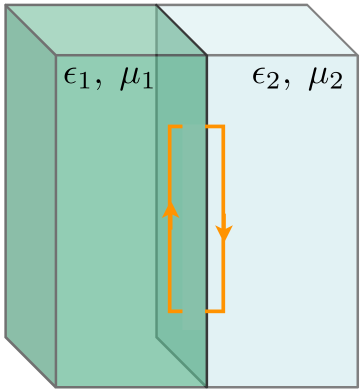
Ampere's law applied to a loop at the interface that has an infintesimal area gives a relationship between the parallel components of the magnetic field. (Note that in most common materials $\mu=\mu_o$)

$$\oint \epsilon E \cdot dA = \sum q$$

$$\oint E \cdot ds = -\frac{d}{dt}\int B \cdot dA$$

$$\oint B \cdot dA = 0$$

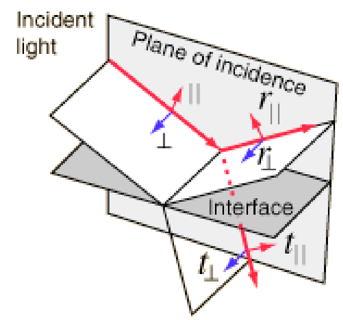
$$\oint \frac{B}{\mu} \cdot ds = \int J \cdot dA + \frac{d}{dt}\int \epsilon E \cdot dA$$

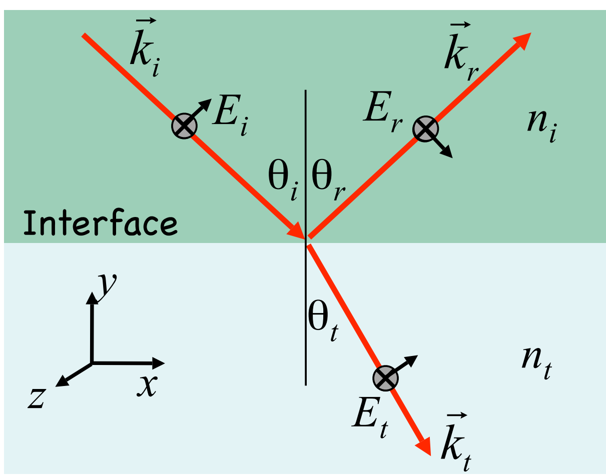



$$\frac{B_{1x,y}}{\mu_1}L - \frac{B_{2x,y}}{\mu_2}L = \int J \cdot dA \,(\to 0) + \frac{d}{dt}\int \epsilon E \cdot dA \,(\to 0) \quad \therefore \quad \frac{B_{1x,y}}{\mu_1} = \frac{B_{2x,y}}{\mu_2}$$

# Reflection at a Boundary

The reflection and transmission coefficients at an interface can be found using the boundary conditions, but they depend on the polarization of the incident light




⊗ "s" polarization (senkrecht, aka TE or vertical) has an E field that is perpendicular to the plane of incidence

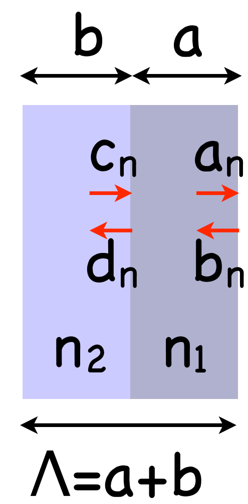
↘ "p" polarization (parallel aka TM or horizontal) has an E field that is parallel to the plane of incidence

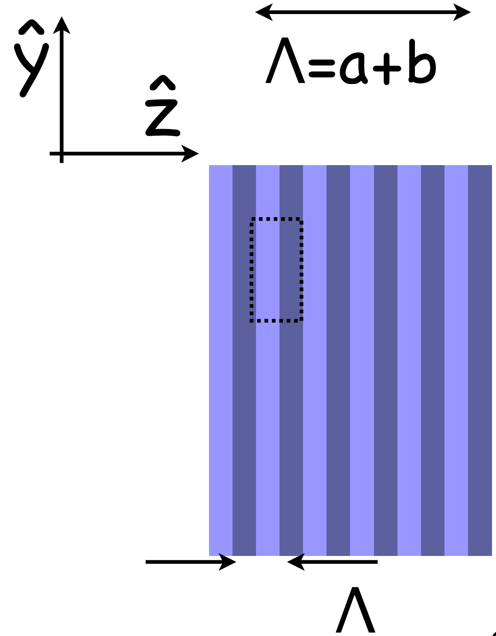
$$D_{1z} = D_{2z}$$
$$E_{1x,y} = E_{2x,y}$$
$$B_{1z} = B_{2z}$$
$$H_{1x,y} = H_{2x,y}$$

# Unit Cell Construct

Label the forwards and backwards going waves in the $n^{th}$ "unit cell"

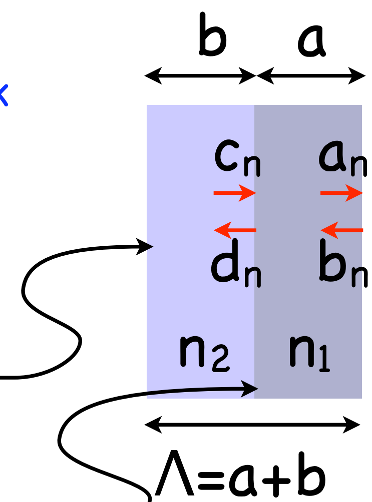
- $a_n$ forward going wave at right side of nth unit cell inside material 1 of form
- $b_n$ backward going wave at right side of nth unit cell inside material 1
- $c_n$ forward going wave at right side of nth unit cell inside material 2
- $d_n$ backward going wave at right side of nth unit cell inside material 2
- a thickness of layer for material 1
- b thickness of layer for material 2
- $\Lambda$ total thickness of unit cell
- n number of unit cells to the right of some arbitrary origin
- forward waves phase factor are expressed as $e^{-ikz+i\omega t}$



# TE Continuity Equations

For TE waves (E is out of the page) $E_x$ and $H_y \propto \partial E_x/\partial z$ must be continuous at each interface




$$
\begin{aligned}
a_{n-1} + b_{n-1} &= c_n e^{ik_{2z}b} + d_n e^{-ik_{2z}b} \\
-ik_{1z}a_{n-1} + ik_{1z}b_{n-1} &= -ik_{2z}c_n e^{ik_{2z}b} + ik_{2z}d_n e^{-ik_{2z}b} \\
c_n + d_n &= a_n e^{ik_{1z}a} + b_n e^{-ik_{1z}a} \\
ik_{2z}c_n - ik_{2z}d_n &= ik_{1z}a_n e^{ik_{1z}a} - ik_{1z}b_n e^{-ik_{2z}a}
\end{aligned}
$$

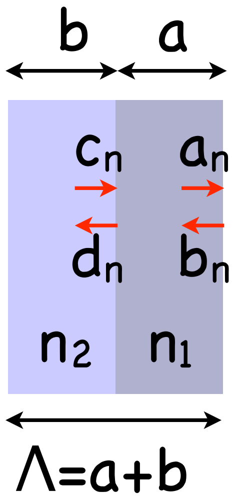
There are 4 equations and 6 unknowns, so these can be manipulated to eliminate $c_n$ and $d_n$ in an expression of the form

$$
\begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}
$$

# Unit Cell Equation (TE)

For TE waves

$$\begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$




with

$$A = e^{ik_{1z}a}\left[\cos(k_{2z}b) + \frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right)\sin(k_{2z}b)\right]$$

$$B = e^{-ik_{1z}a}\left[\frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right)\sin(k_{2z}b)\right]$$

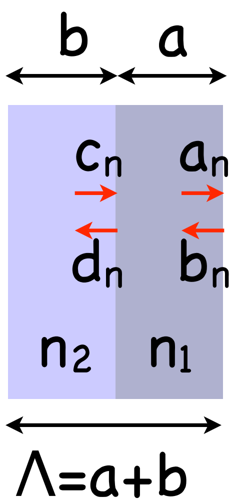
$$C = e^{ik_{1z}a}\left[-\frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right)\sin(k_{2z}b)\right]$$

$$D = e^{-ik_{1z}a}\left[\cos(k_{2z}b) - \frac{i}{2}\left(\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right)\sin(k_{2z}b)\right]$$

# Unit Cell Equation (TM)

For TM waves

$$\begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

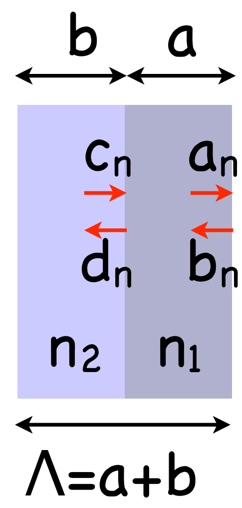



with

$$\begin{aligned}
A &= e^{ik_{1z}a}\left[\cos(k_{2z}b) + \frac{i}{2}\left(\frac{n_1^2 k_{2z}}{n_2^2 k_{1z}} + \frac{n_2^2 k_{1z}}{n_1^2 k_{2z}}\right)\sin(k_{2z}b)\right] \\
B &= e^{-ik_{1z}a}\left[\frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin(k_{2z}b)\right] \\
C &= e^{ik_{1z}a}\left[-\frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin(k_{2z}b)\right] \\
D &= e^{-ik_{1z}a}\left[\cos(k_{2z}b) - \frac{i}{2}\left(\frac{n_1^2 k_{2z}}{n_2^2 k_{1z}} + \frac{n_2^2 k_{1z}}{n_1^2 k_{2z}}\right)\sin(k_{2z}b)\right]
\end{aligned}$$

# Multiple Cell Propagation

From conservation of energy $|a_n|^2 + |b_n|^2 = |a_{n-1}|^2 + |b_{n-1}|^2$ which means the ABCD matrix is "unimodular".




For 2x2 matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{AD - BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

For unimodular matrices the determinant is one, $AD - BC = 1$, so

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

# Bloch Wave Solutions

The propagating waves in the medium are bloch waves with an amplitude that is periodic in Λ and a phase given by Kz, so a bloch wave should obey

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = e^{iK\Lambda} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

requiring $$e^{iK\Lambda} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1}$$

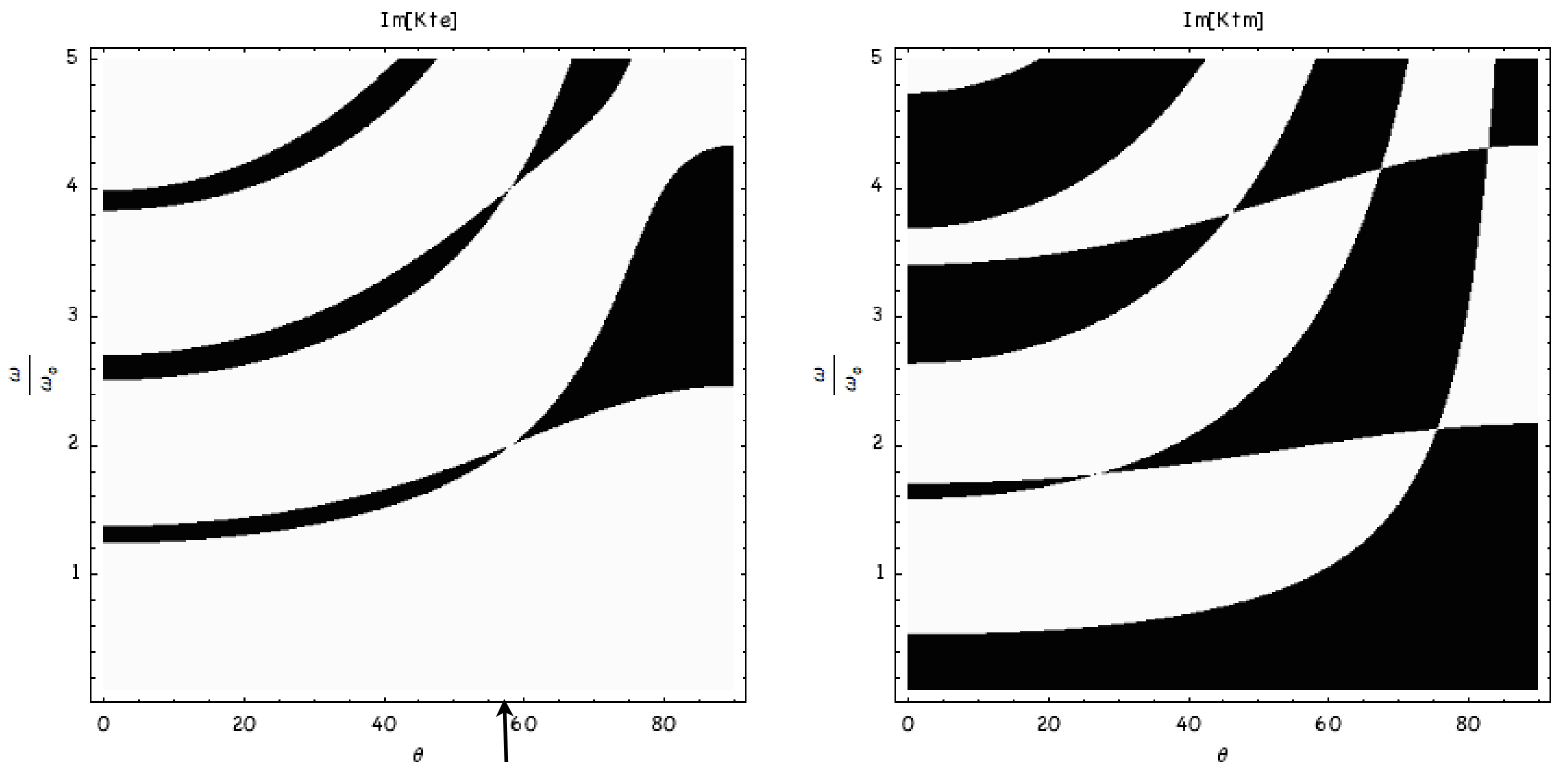
or $$e^{iK\Lambda} = \frac{A+D}{2} \pm i\sqrt{1 - \left(\frac{A+D}{2}\right)^2}$$

with the form $e^{iK\Lambda} = \cos\psi \pm i\sin\psi = e^{\pm i\psi}$ where $\cos\psi = \frac{A+D}{2}$

giving $$K = \frac{1}{\Lambda}\cos^{-1}\left(\frac{A+D}{2}\right)$$

# Material Bandgaps

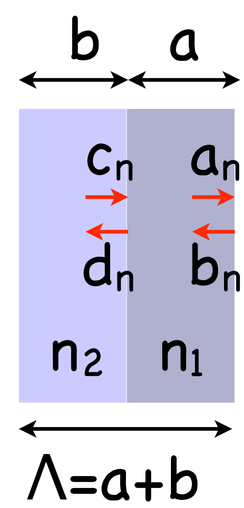
Contour plot Sign[Im[K(θ,ω]]
(stop bands are in black)




Brewster's angle

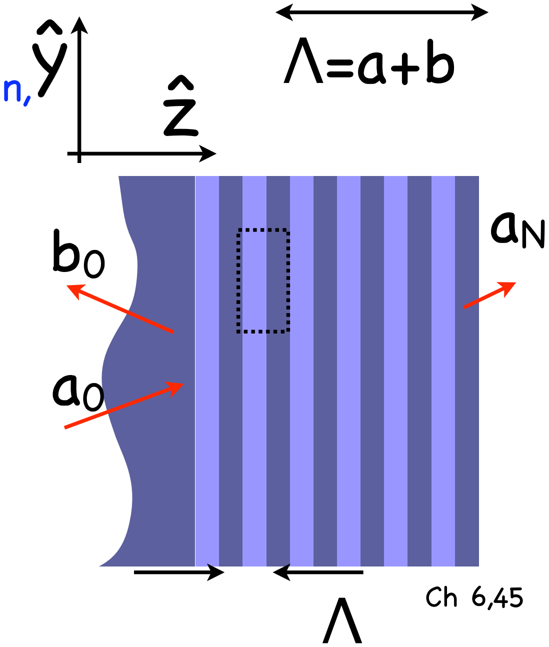
# Bragg Reflection

If a wave is incident on a layered material and cannot propagate because it is within the bandgap, the energy of the wave is reflected.




In the notation where the fields in the nth unit cell of layer 1 are $a_n$, $b_n$, the reflection of a wave $a_0$ incident on the structure in material 1 will have a reflection coefficient,

$$r_N = \frac{b_0}{a_o}$$



# Bragg Reflection

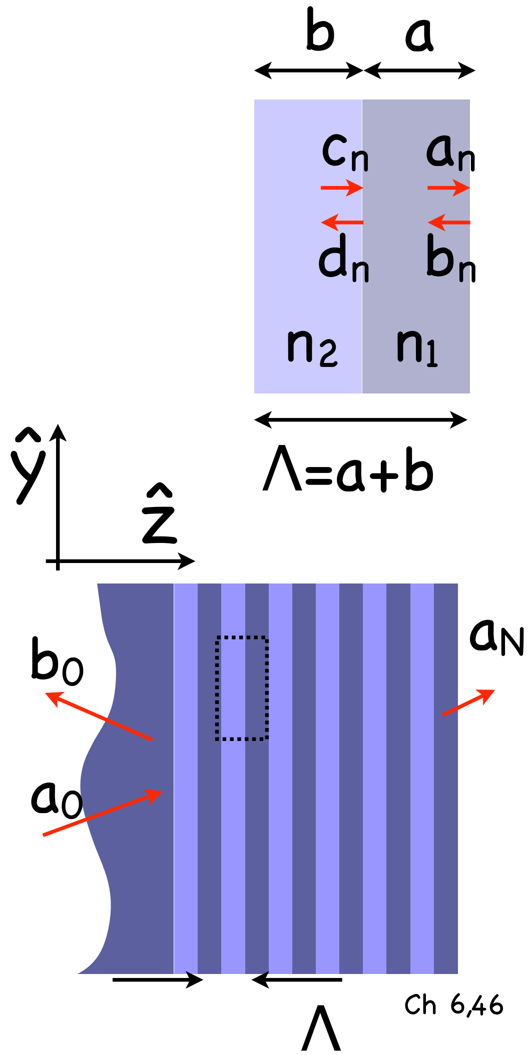
By requiring $b_N=0$, i.e. no input at the far end of the dielectric stack of N layers, we can solve for $b_o$ and $a_o$ in terms of $a_n$.

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N \begin{pmatrix} a_N \\ b_N \end{pmatrix}$$

where

$$\begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

with the values of A, B, C and D previously found.



# Bragg Reflection

For unimodular matrices, Chebyshev' identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^N = \frac{1}{\sin K\Lambda} \begin{pmatrix} A \sin NK\Lambda - \sin{(N-1)K\Lambda} & B \sin NK\Lambda \\ C \sin NK\Lambda & D \sin NK\Lambda - \sin{(N-1)K\Lambda} \end{pmatrix}$$

with $K\Lambda = \cos^{-1}\left(\frac{A+D}{2}\right)$

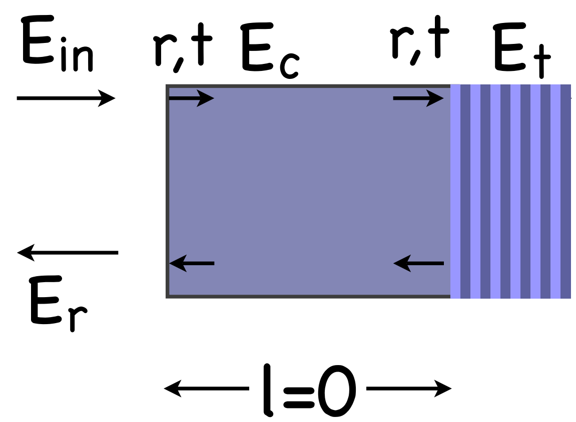
so

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \frac{1}{\sin K\Lambda} \begin{pmatrix} A \sin NK\Lambda - \sin{(N-1)K\Lambda} & B \sin NK\Lambda \\ C \sin NK\Lambda & D \sin NK\Lambda - \sin{(N-1)K\Lambda} \end{pmatrix} \begin{pmatrix} a_n \\ 0 \end{pmatrix}$$

$$r_N = \frac{b_0}{a_o} = \frac{C \sin NK\Lambda}{A \sin NK\Lambda - \sin(N-1)K\Lambda}$$

# Structure Reflectivity in Air

Consider an infintesimal thickness of material 1 on top of the layered stucture. It has reflectivity $r_{a1}$ on the air side (from air to material 1), and reflectivity $r_N$ on the structure side.




$$E_c = tE_{in} - r_{a1} r_N E_c \quad \text{and} \quad E_r = r_{a1} E_{in} + r_N t_{a1} E_c$$

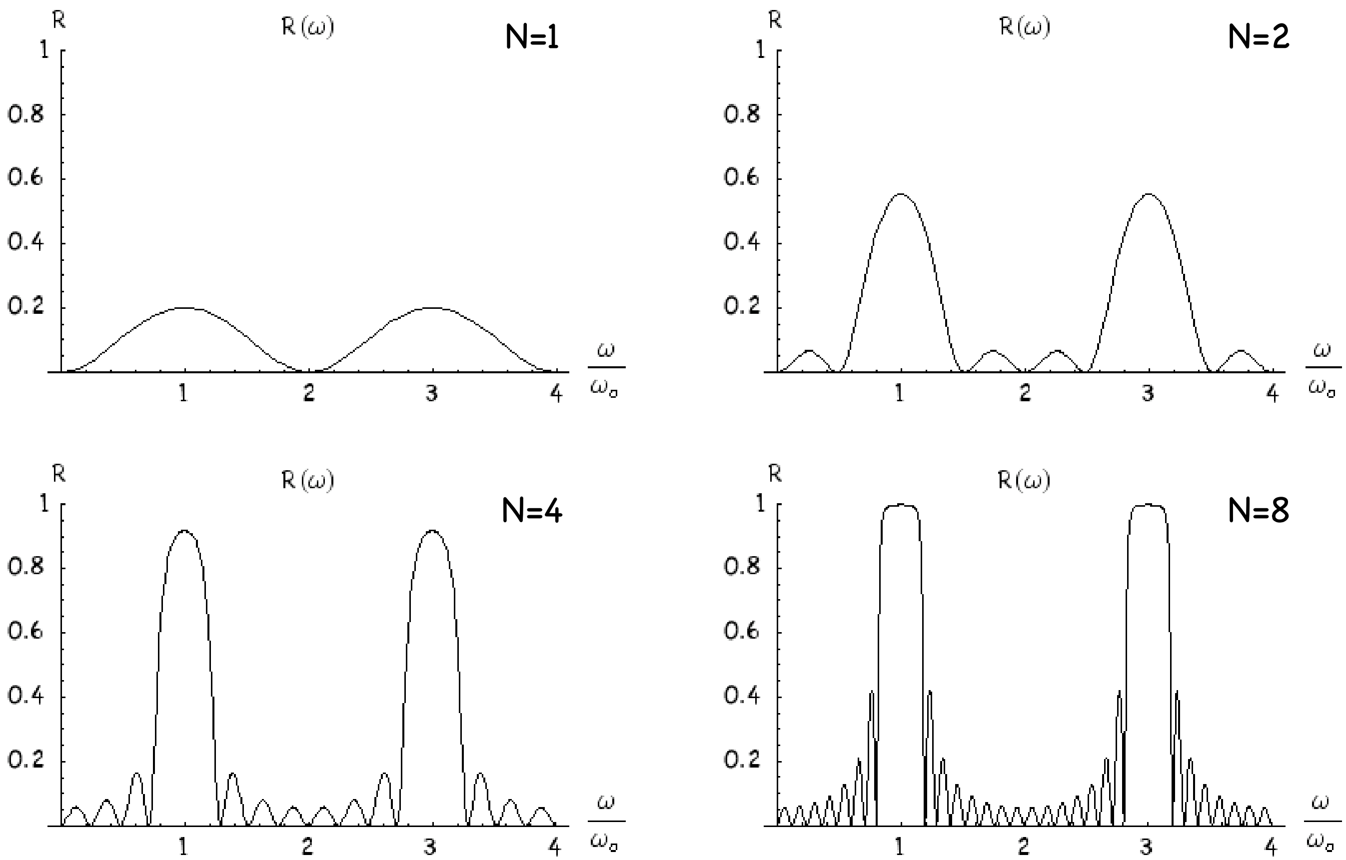
giving $$E_c = \frac{tE_{in}}{1 + r_{a1} r_N}$$ and $$E_r = \left( r_{a1} + \frac{t_{a1}^2}{1 + r_{a1} r_N} \right) E_{in}$$

so the reflectivity of the structure in air is $$r = \frac{r_{a1} + r_N}{1 + r_{a1} r_N}$$

# Spectral Reflectivity

Spectral reflectivity $R_N$ at normal incidence of an N layer stack (quarter wave at $\omega_0$, $n_h$=2.5, $n_l$=1.5)

$$R_n = |r_N|^2 = \left| \frac{C \sin NK\Lambda}{A \sin NK\Lambda - \sin(N-1)K\Lambda} \right|^2$$



# Summary

# References

- Yariv & Yeh "Optical Waves in Crystals" chapter 6
- http://www.tf.uni-kiel.de/matwis/amat/semi_en/kap_2/backbone/r2_1_4.html