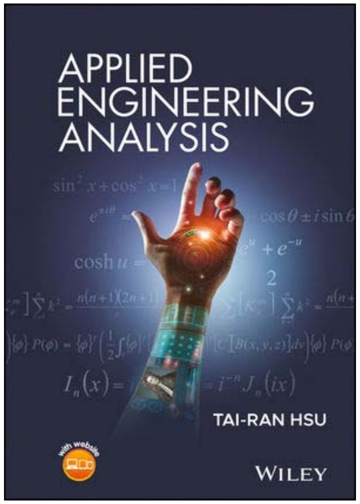
Applied Engineering Analysis
- slides for class teaching*

Chapter 9

Application of Partial Differential Equations in Mechanical Engineering Analysis

* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, published by John Wiley & Sons, 2018 (ISBN 9781119071204)



Chapter Learning Objectives

- Learn the physical meaning of partial derivatives of functions.

- Learn that there are different order of partial derivatives describing the rate of changes of functions representing real physical quantities.

- Learn the two commonly used technique for solving partial differential equations by (1) Integral transform methods that include the Laplace transform for physical problems covering half-space, and the Fourier transform method for problems that cover the entire space; (2) the "separation of variable technique."

- Learn the use of the separation of variable technique to solve partial differential equations relating to heat conduction in solids and vibration of solids in multidimensional systems.

9.1 Introduction

A partial differential equation is an equation that involves partial derivatives.

Like ordinary differential equations, Partial differential equations for engineering analysis are derived by engineers based on the physical laws as stipulated in Chapter 7.

Partial differential equations can be categorized as "*Boundary-value problems*" or "*Initial-value problems*", or "*Initial-boundary value problems*":

(1) The *Boundary-value problems* are the ones that the complete solution of the partial differential equation is possible with specific boundary conditions.

(2) The *Initial-value problems* are those partial differential equations for which the complete solution of the equation is possible with specific information at one particular instant (i.e., time point)

Solutions to most these problems require specified both boundary and initial conditions.

9.2 **Partial Derivatives** **(p.285)**:

A partial derivative represents the rate of change of a function involving more than one variable (2 in minimum and 4 in maximum). Many physical phenomena need to be defined by more than one variable as in the following instance:

Example of partial derivatives: The ambient temperatures somewhere in California depend on where and where this temperature is counted. Therefore, the magnitude of the temperature needs to be expressed in mathematical form of T(x,y,z,t), in which the variables x, y and z in the function T indicate the location at which the temperature is measured and the variable t indicates the time of the day or the month of the yaer at which the measurement is taken. The rate of change of the magnitude of the temperature, i.e., the derivatives of the function T(x,y,z,t) needs to be dealt with the change of *EACH* of all these 4 variables accounted with this function. In other words, we may have all together 4 (not just one) such derivatives to be considered in the analysis. Each of these 4 derivative is called "*partial derivative*" of the function T(x,y,z,t) because each derivative as we will express mathematically can only represent "*part*" (not whole) of the derivative for this function that involves multi-variables.

There are two kinds of independent variables in partial derivatives:

(1) "*Spatial*" variables represented by (x,y,z) in a rectangular coordinate system, or (r,θ,z) in a cylindrical polar coordinate system, and

(2) The "*Temporal*" variable represented by time, t.

9.2 **Partial Derivatives**: - Cont'd

Mathematical expressions of partial derivatives (p.286)

We have learned from Section 2.2.5.2 (p.33) that the derivative for function with only *one* variable, such as f(x) can be defined mathematically in the following expression, with physical meaning shown in Figure 9.1.:

$$\frac{df(x)}{dx} = \ell im_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (2.9)$$

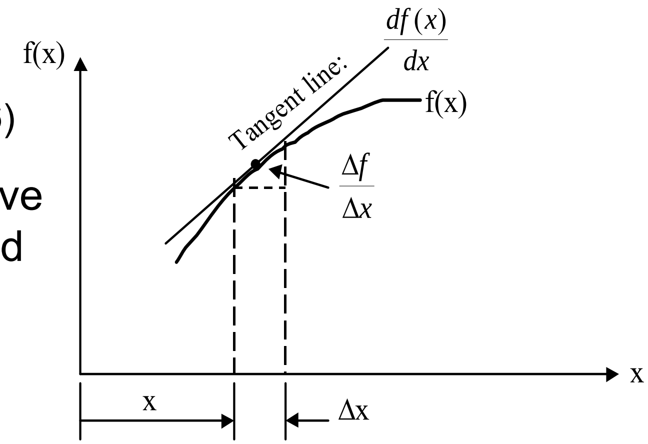


Figure 9.1

For functions involving with more than one independent variable, e.g. x and t expressed in function f(x,t), we need to express the derivative of this function with *BOTH* of the independent variables x and t separately, as shown below:

The partial derivative of function f(x,t) with respect to x only may be expressed in a similar way as we did with function f(x) in Equation (2.9), or in the following way:

$$\frac{\partial f(x,t)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,t) - f(x,t)}{\Delta x} \qquad (9.1)$$

We notice that we treated the other independent variable t as a "constant" in the above expression for the partial derivative of function f(x,t) with respect to variable x .

Likewise, the derivative of function f(x,t) with respect to the other variable t is expressed as:

$$\frac{\partial f(x,t)}{\partial t} = \ell im_{\Delta t \to 0} \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} \qquad (9.2)$$

9.2 Partial Derivatives: - Cont'd

Mathematical expressions of higher orders of partial derivatives:

Higher order of partial derivatives can be expressed in a similar way as for ordinary functions, such as:

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\dfrac{\partial f(x+\Delta x,t)}{\partial x} - \dfrac{\partial f(x,t)}{\partial x}}{\Delta x} \tag{9.3}$$

and

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \lim_{\Delta t \to 0} \frac{\dfrac{\partial f(x,t+\Delta t)}{\partial t} - \dfrac{\partial f(x,t)}{\partial t}}{\Delta t} \tag{9.4}$$

There exists another form of second order partial derivatives with cross differentiations with respect to its variables in the form: $\dfrac{\partial^2 f(x,t)}{\partial x \partial t} = \dfrac{\partial^2 f(x,t)}{\partial t \partial x}$ (9.5)

9.3 Solution Methods for Partial Differential Equations (PDEs) (p.287)

There are a number ways to solve PDEs analytically; Among these are: (1) using integral transform methods by "transforming one variable to parametric domain after another in the equations that involve partial derivatives with multi-variables. Fourier transform and Laplace transform methods are among these popular methods. The recent available numerical methods such as the finite element method, as will present in Chapter 11 offers much practical values in solving problems involving extremely complex geometry and prescribed physical conditions. The latter method appears having replaced much effort required in solving PDEs using classical methods. With readily available digital computers and affordable commercial software such and ANSYS code, this method has been widely accepted by industry. The classical solution methods appears less in demand in engineering analysis as time evolves.

9.3 **Solution Methods for Partial Differential Equations-**Cont'd

9.3.1 The separation of variables method (p.287):

The essence of this method is to "***separate***" the independent variables, such as x, y, z, and t involved in the functions and partial derivatives appeared in the PDEs.

We will illustrate the principle of this solution technique with a function F(x,y,t) in a partial differential equation. The process begins with an assumption of the original function F(x,y,t), to be a product of three functions, each involves only one of the three independent variables, as expressed in Equation (9.6), as shown below:

$$F(x,y,t) = f_1(x)f_2(y)f_3(t) \tag{9.6}$$

where $f_1(x)$ is a function of variable x only
$f_2(y)$ is a function of variable y only, and
$f_3(t)$ is a function of variable t only

Equation (9.6) has effectively *separated* the three independent variables in the original function F(x,y,t) into the *product of three separate functions*; each consists of only *one* of the three independent variables.

The 3 separate function f_1, f_2 and f_3 in Equation (9.6) will be obtained by solving 3 individual *ordinary differential equations* involving "separation constants." We may than use the methods for solving ordinary differential equations learned in Chapters 7 and 8 to solve these 3 ordinary differential equations.

The partial differential equation that involve the function F(x,y,t) and its partial derivatives can thus be solved by equivalent ordinary differential equations via the separation relationship shown in Equation (9.6) . In general, PDEs with n independent variables can be separated into n ordinary differential equations with (n-1) separation constants. The number of required given conditions for complete solutions of the separated ordinary differential equations is equal to the orders of the separated ordinary differential equations.

9.3 **Solution Methods for Partial Differential Equations-**Cont'd

9.3.2 Laplace transform method for solution of partial differential equations (p.288):

We have learned to use Laplace transform method to solve ordinary differential equations in Section 6.6, in which the only variable, say "x", involved with the function in the differential equation y(x) must cover the half space of (o<x<∞). Solution of the differential equation y(x) is obtained by converting this equation into an algebraic equation by Laplace transformation with the "transformed expression F(s) in which "s" is the Laplace transform parameter. The solution of the ordinary differential equation y(x) is obtained by inverting the F(s) in its resulting expression. We have also use the Laplace transform method to solve a partial differential equation in Example 6.19 (p.194) after having learned how to transform partial derivatives in Section 6.7.

9.3.3 Fourier transform method for solution of partial differential equations (p.288):

Fourier transform engineering analysis needs to satisfy the conditions that the variables that are to be transformed by Fourier transform should cover the entire domain of (-∞, ∞). Mathematically, it has the form:

$$\Im[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = F(\omega) \qquad (9.7)$$

The inverse Fourier transform is:

$$\Im^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \qquad (9.8)$$

The following Table 9.1 presents a few useful formula for Fourier transforms of a few selected functions.

Functions for Fourier Transform f(x)	After Fourier Transform F(ω)
(1) f(x-a)	$F(\omega)e^{-i\omega a}$
(2) δ(x)*	1
(3) u(x)*	$(i\omega)^{-1}$
(4) $e^{-\alpha\lvert x\rvert}$ $\alpha > 0$	$2a/(a^2 - \omega^2)$
(5) u(x)sinax	$a/(a^2 + \omega^2)$
(6) u(x)cosax	$i\omega/(a^2 - \omega^2)$

*δ(x) = Delta function, or impulsive function and u(x) is the unit step function. Both these functions are defined in Section 2.4.2

9.3.3 Fourier transform method for solution of partial differential equations:-Cont'd

Example 9.2

Solve the following partial differential equation using Fourier transform method.

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \alpha^2 \frac{\partial T(x,t)}{\partial t} \qquad -\infty < x < \infty \tag{9.11}$$

where the coefficient α is a constant. The equation satisfies the following specified condition:

$$T(x,t)\big|_{t=0} = T(x,0) = f(x) \tag{9.12}$$

Solution

We will transform variable x in the function T(x,t) in Equation (9.11) using Fourier transform in Equation (9,7):

$$T^*(\omega,t) = \Im[T(x,t)] = \int_{-\infty}^{\infty} T(x,t)e^{-i\omega x}dx \tag{a}$$

Apply the above integral to the left-hand-side of Equation (9.11) will yield:

$$\Im\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] = \int_{-\infty}^{\infty}\left(\frac{\partial^2 T(x,t)}{\partial x^2}\right)e^{-i\omega x}dx = -\omega^2 T^*(\omega,t)$$ from Equation (9.10), and

$$\Im\left[\alpha^2\frac{\partial T(x,t)}{\partial t}\right] = \alpha^2\int_{-\infty}^{\infty}\left(\frac{\partial Tx,t}{dt}\right)e^{-i\omega x}dx = \alpha^2\frac{\partial}{\partial t}\int_{-\infty}^{\infty}T(x,t)e^{-i\omega x}dx = \alpha^2\frac{\partial T^*(\omega,t)}{\partial t}$$ for the right-hand-side of Eq. (9.11)

Equation (9.11) has the form after the transformation:

$$-\omega^2 T^*(\omega,t) = \alpha^2\frac{dT^*(\omega,t)}{dt} \tag{b}$$

Equation (b) is a first order ordinary differential equation involving the function T*(ω,t) and the method of obtaining the general solution of this equation is available in Chapter 7.

At this point, we need to transform the specified condition in Equation (9.12) by the Fourier transform defined in Equation (a), or by the following expression:

$$T^*(\omega,0) = \Im[T(x,0)] = \int_{-\infty}^{\infty}T(x,0)e^{-i\omega x}dx = \int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx = g(\omega) \tag{c}$$

9.3.3 Fourier transform method for solution of partial differential equations:-Cont'd

Example 9.2- Cont'd

We will solve the first order ODE in Equation (b) with the solution of T*(ω,t) in Equation (b) and obtain:

$$T^*(\omega,t) = g(\omega)e^{-\frac{\omega^2}{\alpha^2}t} \tag{d}$$

The solution of the partial differential equation in Equation (9.11) with the specified condition in Equation (9.12) can thus be obtained by inverting the transform T*(ω,t) to T(x,t) using Equation (9.8) by the following expression:

$$T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^*(\omega,t)e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}[g(\omega)]e^{-\left(\frac{\omega}{\alpha}\right)^2 t}e^{i\omega x}d\omega \tag{e}$$

where g(ω) is available in Equation (c) to be the Fourier transformed specified condition of T(x,0) in Equation (9.12).

9.4 Partial Differential Equations for Heat Conduction in Solids (p.291)

9.4.1 Heat conduction in engineering analysis

We have learned from Section 7.5 (p.217) that temperature variations in media is induced by heat transmissions. This variation of temperature in media (solids or fluids) is called temperature field.

Heat transfer is a very important branch of mechanical and aerospace engineering analyses because many machines and devices in both these engineering disciplines are vulnerable to heat. According to statistics, over 60% of electronics devices in the US Airforce failed to functions due to excessive heating. Excessive heat flow can also result in a high temperature fields in the structural media, which may result in serious thermal stresses in addition to significant deterioration of material strength and property changes, as presented in Section 7.5.

In this section, we will derive the partial differential equations for heat conduction in solids in both rectangular and cylindrical polar coordinate systems, and solve these equations by using separations of variables technique. Although many of these problems can also be solved by advanced numerical techniques such as finite difference and finite element methods, the classic solutions as will be presented in this chapter, however, will offer engineers with solutions at anywhere in the solid structure, which the numerical methods cannot offer the same. These numerical methods, however, are often used for situations that involve complicated geometry, loading and boundary conditions.

9.4.2 Derivation of partial differential equations for heat conduction analysis

Heat conduction equation is used to determine the temperature distributions induced by heat conduction in solids, either by heat generation by the solids or by heat from external sources.

This equation will be derived from the law of conservation of energy, in particular, the first law of thermodynamics.

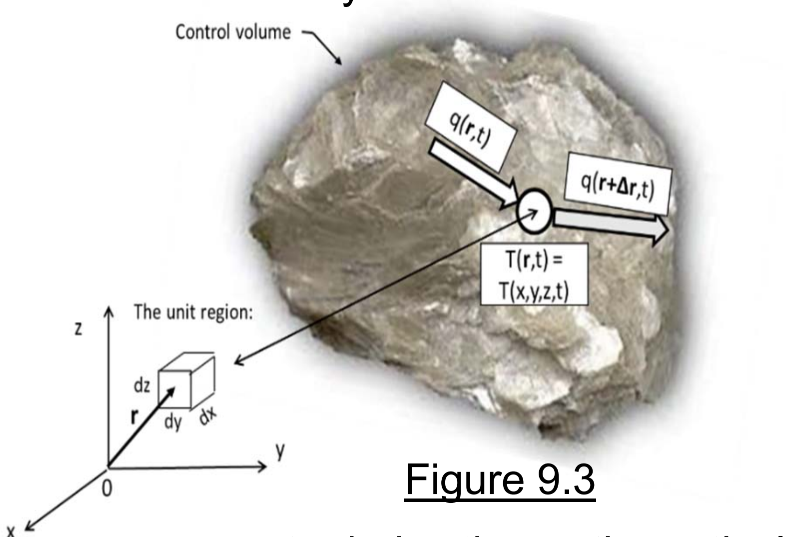


Figure 9.3

By referring to Figure 9.3 , a solid with a volume is subjected to heat flow in the form of heat flux **q**(**r**,t) from external sources to a small element (in the small open circle) in the figure.

The heat leaving the element is **q**(**r+Δr**,t) with **r** designating the spatial variables of (x,y,z) in a rectangular coordinate system or (r,θ,z) in a cylindrical polar coordinate system. Since heat is a form of energy, we may use the law of conservation of energy in the following block diagrams to derive the mathematical expression for the case:

Rate of heat entering the solid element	+	**Rate of heat generation by the solid element**

= | **Rate of heat leaving the solid element** | + | **Rate of energy storage in the solid element** |

9.4.2 **Derivation of partial differential equations for heat conduction analysis –** Cont'd

We may use the following mathematical expressions to represent the physical quantities in the solid shown in Figure 9.3.

Heat fluxes entering and leaving the small element in Figure 9.3 may be expressed by the Fourier law of heat conduction in the following forms:

$$\boldsymbol{q}(\boldsymbol{r},t)=\mp k\nabla T(\boldsymbol{r},t)$$

The energy storage in the element = change of internal energy: $\Delta u = \rho c \Delta T$, in which ρ=mass density, c=specific heat Of the solid and ΔT=temperature rise or fall in the solid

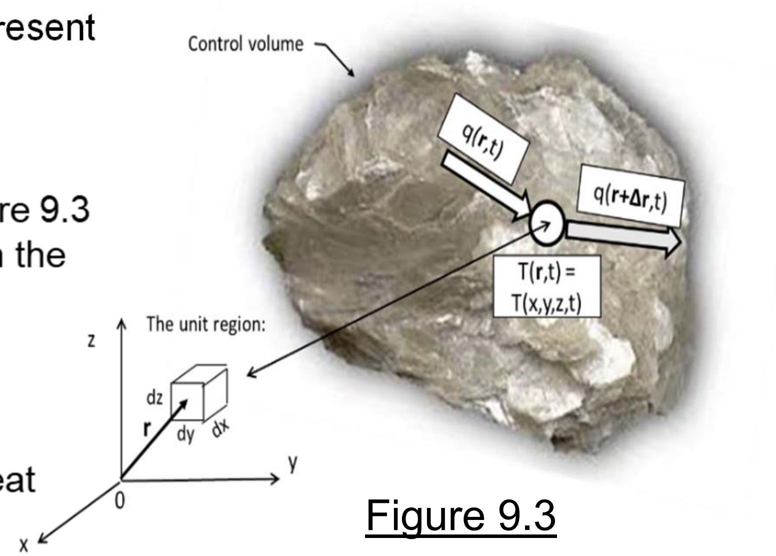


Figure 9.3

From the block diagram of energy conservation and the above mathematical representations of physical quantities in the block diagram, we may establish the following partial differential equation for the temperature variations in the entire solid to be:

$$\rho c \frac{\partial T(\mathbf{r},t)}{\partial t} = \nabla \bullet [k\nabla T(\mathbf{r},t)] + Q(\mathbf{r},t) \qquad (9.13)$$

where k = thermal conductivity of the solid material, Q(**r**,t)= heat generation by the material (such as Ohm heating of $Q=iR^2$ with i being the electric current in Ampere, and R is the electric resistance of the material in Ohms.

9.4.3 Heat conduction equation in rectangular coordinate system

The general heat conduction equation in Equation (9.13) will take the following form with T(**r**,t) = T(x,y,z,t):

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k_x\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial z}\left[k_z\frac{\partial T}{\partial z}\right]+Q(x,y,z,t) \qquad (9.14a)$$

in which k_x, k_y and k_z are the thermal conductivities of the solid along the x-, y- and z-coordinates respectively.

9.4.4 Heat conduction equation in cylindrical polar coordinate system:

Heat conduction equation in this coordinate system is obtained by expanding Equation (9.8) as follows with T(**r**,t) = T(r,θ,z,t):

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial r}\left[k_r\frac{\partial T}{\partial r}\right]+\frac{1}{r}\left[k_r\frac{\partial T}{\partial r}\right]+\frac{1}{r^2}\left[\frac{\partial}{\partial \theta}k_\theta\frac{\partial T}{\partial \theta}\right]+\frac{\partial}{\partial z}\left[k_z\frac{\partial T}{\partial z}\right]+Q(r,\theta,z,t) \qquad (9.14b)$$

where k_r,k_θ and k_z are thermal conductivities of the material along the r-, θ- and z-coordinate respectively.

9.4.5 General heat conduction equation (p.293):

Thermal conductivities k_x, k_y and k_z in Equation (9.14a) and k_r, k_θ and k_z in Equation (9.14b) are used for heat conduction analysis of solids with their thermophysical properties varying in different directions, such as for fiber filament composites. For most engineering analyses, such variation of thermophysical properties do not exist. Consequently a generalize heat conduction equation may be expressed as follows

$$\nabla^2 T(\mathbf{r},t) + \frac{Q(\mathbf{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t} \tag{9.15}$$

where k = thermal conductivity of the material and Q(**r**,t) is the heat generated by the material per unit volume and time.

The symbol α in Equation (9.15) is "thermal diffusivity" of the material with its value equals to: $\alpha = \dfrac{k}{\rho c}$, it is often used as a measure on how "fast" heat can flow by conduction in solids.

9.4.6 Initial conditions:

Complete solution of heat conduction equation in Equation (9.15) involves determining a number of arbitrary constants according to specific initial and boundary conditions.
These conditions are necessary to translate the real physical conditions into mathematical expressions.
Initial conditions are required only when dealing with transient heat transfer problems in which temperature field in a solid changes with elapsing time. The common initial condition in a solid can be expressed mathematically as:

$$T(\mathbf{r},t)\big|_{t=0} = T(\mathbf{r},0) = T_0(\mathbf{r}) \tag{9.16}$$

where the temperature field T_0(**r**) is a specified function of the spatial coordinates **r** only

In many practical applications, the initial temperature distribution T_0(**r**) in Equation (9.16) can be assigned with a constant value such as room temperature at 20°C for a uniform temperature condition in the solid.

9.4.6 **Boundary conditions:**

Specific boundary conditions are required in obtaining complete solutions in heat transfer analyses using the general heat conduction equation in (9.15). Four types of boundary conditions are available for this purposes.as will be presented below.

1) Prescribed surface temperature, $T_s(t)$:

This type of boundary condition is used to have the temperature at the surface of the solid structure measured by either attaching thermocouples to the structure surface or by some non-contact methods such as infrared thermal imaging scanning camera. The mathematical expression for this case takes the form:

$$T(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r}_s} = T_s(t) \tag{9.17a}$$

where $\mathbf{r_s}$ is the coordinates of the boundary surface where temperature are specified to be $T_s(t)$

2) Prescribed heat flux boundary condition, $q_s(t)$:

Many structures have their surfaces exposed to a heat source or a heat sink, in such situations, heat is being supplied to or removed from the solids through its outside surface. The mathematical translation of the heat flux to or from a solid surface can be readily carried out by using the Fourier law of heat conduction defined in Equation (7.25). The mathematical formulation of the heat flux across a solid boundary surface can be expressed as:

$$\left.\frac{\partial T(\mathbf{r},t)}{\partial \mathbf{n_i}}\right|_{\mathbf{r}=\mathbf{r_s}} = -\frac{q_s(\mathbf{r_s},t)}{k} \tag{9.17b}$$

where k is the thermal conductivity of the solid material. The symbol $\frac{\partial}{\partial \mathbf{n_i}}$ is the differentiation along the outward-drawn normal to the boundary surface S_i. We may express Equation (9.17b) for the boundaries that are impermeable to heat flow, or a boundary that is thermally insulated as:

$$\left.\frac{\partial T(\overline{\mathbf{r}},t)}{\partial \mathbf{n}}\right|_{\mathbf{r}=r_s} = 0 \tag{9.17c}$$

9.4.6 **Boundary conditions – Cont'd:**

3) Convective boundary conditions:

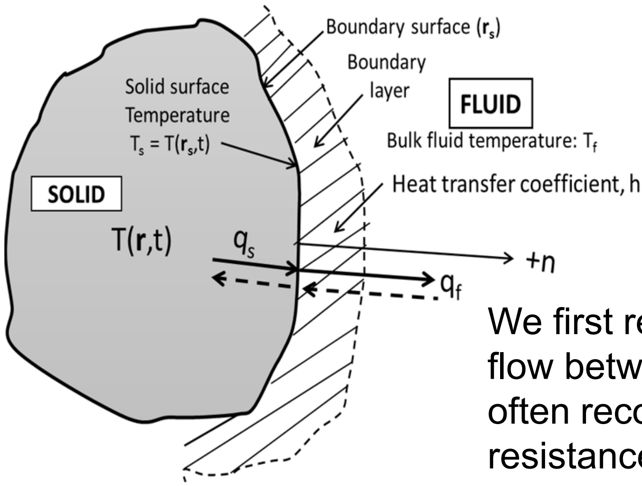


Figure 9.5

This type of boundary condition applies when the solid structure is either in contact with a fluid, or is submerged in fluids, as often happen in reality.

Let us derive the mathematical expressions of the boundary conditions by referring to the sketch in Figure 9.5.

We first recognize that there is a physical "barrier" that retards free heat flow between the solid surface and its contacted fluid. This barrier is often recognized as the "boundary layer that can be characterized by a "film resistance that is equal to "1/h" with h being the film coefficient as defined in Equation (7.29) in Section 7.5.5. Physically it means that the temperature of the solid surface $T_s \neq$ the temperature of the surrounding bulk fluid T_f.

The following two (2) mathematical expressions are derived to represent the above physical phenomenon: From the fact that no heat is being stored at the interface of the solid and fluid, which leads to the following Equality:

$\boxed{\text{Heat flow in solid}} = \boxed{\text{Heat flow in fluid}}$ $\Rightarrow$ $-k\left.\dfrac{\partial T(\mathbf{r},t)}{\partial n}\right|_{\mathbf{r}=\mathbf{r}_s} = h\left[T(\mathbf{r}_s,t)-T_f\right]$ or in the form:

$$\left.\frac{\partial T(\mathbf{r},t)}{\partial n}\right|_{\mathbf{r}=r_s} + \frac{h}{k}T(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r}_s} = \frac{h}{k}T_f \qquad (9.17d)$$

The above equation involves heat flows in solids by conduction and heat flows in fluids by convection. It is often referred to be the "mixed boundary conditions." This expression of boundary condition actually could be used for problems involving prescribed surface temperatures in Equation (9.17a) with h→∞, We may also prove that letting h = 0 in Equation (9.17d) will lead to a thermally insulated boundary condition with q_s = 0 in Equation (9.17b).

Example 9.3 (p.295)

Show the appropriate boundary conditions of a long thick wall pipe containing hot steam flow inside the pipe at a bulk temperature T_s with heat transfer coefficient h_s.

The outside wall of the pipe is in contact with cold air at a temperature of T_a and with a heat transfer coefficient h_a, as illustrated in Figure 9.6.

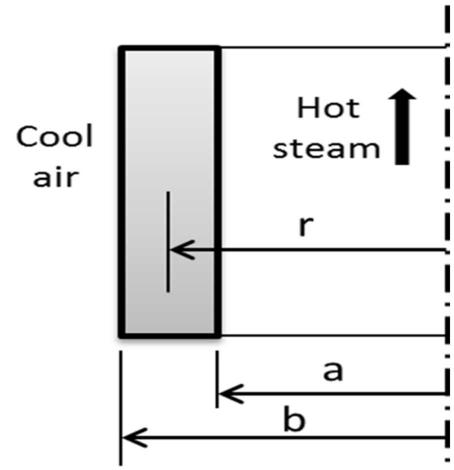


Figure 9.6

Solution

A common but logical hypothesize made in this type of engineering analysis is that heat will flow is primarily along the positive radial direction (r) in a long pipe such as in this example because of the greater temperature gradient cross the pipe wall than that along the length. So, the radial direction is the principal direction of heat flow. Consequently, we will account for two boundary surfaces in this analysis, i.e., at the inner surface with r = a and the outside surface at r=b.

Since heat transfer coefficients of both the steam inside the pipe (h_s) and the heat transfer coefficient of the air outside the pipe (h_a) are given, we may use Equation (9.17d) to establish the convective boundary conditions at both sides of the pipe wall as follows:

(a) At inner boundary with r = a:

$$k\frac{dT(r)}{dr}\bigg|_{r=a} - \frac{h_s}{k}T(r)\bigg|_{r=a} = \frac{h_s}{k}T_s$$

(b) At the outside boundary with r = b:

$$k\frac{dT(r)}{dr}\bigg|_{r=b} + \frac{h_a}{k}T(r)\big|_{r=b} = \frac{h_a}{k}T_a$$

in which k = thermal conductivity of the pipe material

Example 9.4 (p.296)

Find the temperature distribution in a long thick wall pipe with inner and outside radii a and b respectively by using the three types of boundary conditions in Equations (9.17a,b,d).
Conditions for establishing the mathematical expressions for these boundary conditions with hot steam inside the pipe and the cool surrounding air outside the pipe are indicated in Figure 9.7.

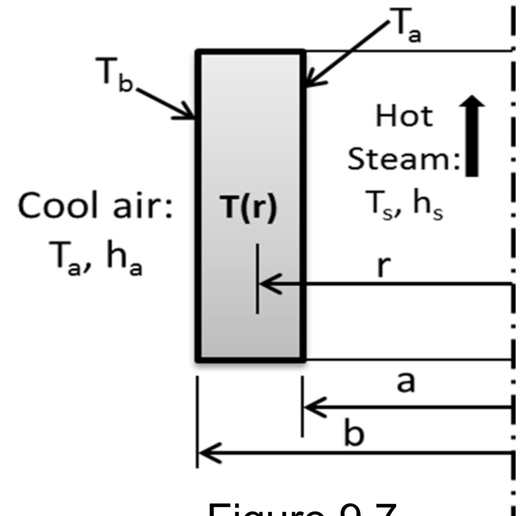


Figure 9.7

Solution

We adopt the same principal as described in the last example that the shorter heat flow path along the radial direction of the pipe enables us to assume the principal temperature variation in the pipe wall is with the radius variable (r). Consequently, we may assume that the temperature function that we desire in this analysis is T(r) **only**.
Thus, by select the relevant terms in the PDE in (9.14b), we will have the relevant differential equation of the form:

$$\frac{d^2T(r)}{dr^2}+\frac{1}{r}\frac{dT(r)}{dr}=0 \qquad \text{(a)}$$

Solution of the differential equation in (a) may be obtained by either using Equation (8.6), or by re-arranging the terms that fit the following form of:

$$\frac{d}{dr}\left[r\frac{dT(r)}{dr}\right]=0 \qquad \text{(b)}$$

from which we get the solution T(r) by integrating Equation (b) twice with respect to variable r, leading to the form:

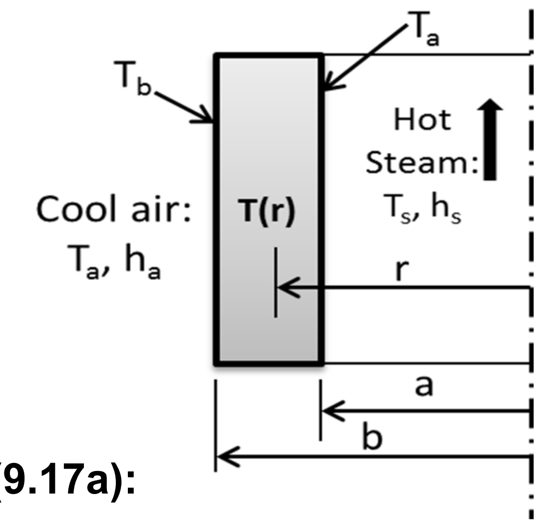
$$T(r)=c_1\,\ell n(r)+c_2 \qquad \text{(c)}$$

where c_1 and c_2 are two arbitrary constants

Example 9.4-Cont'd

We have derived the general solution of the temperature across the pipe wall to be:

$$T(r) = c_1 \ell n(r) + c_2 \quad \text{(c)}$$



We will determine the two arbitrary constants c_1 and c_2 using the 3 different sets of boundary conditions presented in Section 9.4.6 as follows:

(A) With prescribed boundary conditions in Equation (9.17a):

With the given conditions of: T_a to be the temperature at the inner surface with $T(a) = T_a$, and $T(b) = T_b$ at the outside surface of the pipe, we will determine the two constants in Equation (c) to be:

$$c_1 = \frac{T_a - T_b}{\ell n\left(\frac{a}{b}\right)} \quad and \quad c_2 = T_a - \frac{T_a - T_b}{\ell n\left(\frac{a}{b}\right)} \ell n(a)$$

which leads to the following complete solution:

$$T(r) = T_a - \frac{T_a - T_b}{\ell n\left(\frac{a}{b}\right)} \ell n\left(\frac{r}{a}\right) \quad \text{(d)}$$

(B) With prescribed heat flux q_a across the inner surface and T_b at the outside surface:

at inner surface: $$\left.\frac{dT(r)}{dr}\right|_{r=a} = -\frac{q_a}{k} \quad \text{(e)}$$

at outside surface: $$\left.T(r)\right|_{r=b} = T(b) = T_b \quad \text{(f)}$$

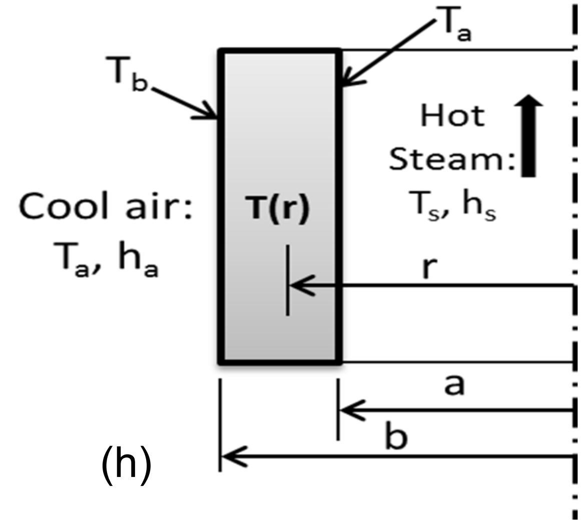
We may determine the constants c_1 and c_2 in Equation (c) to be: $c_1 = -\frac{aq_a}{k}$ and $c_2 = T_b + \frac{aq_a}{k} \ell n(b)$

which leads to the comolete solution of Equation (a) to be:

$$T(r) = T_b - \frac{aq_a}{k} \ell n\left(\frac{r}{b}\right) \quad \text{(g)}$$

Example 9.4-Cont'd

(c) With mixed boundary conditions in Equation (9.17d):



The 2 appropriate boundary conditions are:

at inner pipe surface: $$\left.\frac{dT(r)}{dr}\right|_{r=a} - \frac{h_s}{k}T(r)\Big|_{r=a} = \frac{h_s}{k}T_s \qquad \text{(h)}$$

at outside pipe surface: $$\left.\frac{dT(r)}{dr}\right|_{r=b} + \frac{h_a}{k}T(r)\Big|_{r=b} = \frac{h_a}{k}T_a \qquad \text{(j)}$$

Substitute (h) and (j) into Equation (c), we will get:

$$c_1 = \frac{h_s h_a (T_a - T_s)}{\frac{kh_a}{a} + \frac{kh_s}{b} + h_s h_a \ell n\left(\frac{b}{a}\right)}$$

and

$$c_2 = T_a - \frac{h_s h_a (T_a - T_s)}{\frac{kh_a}{a} + \frac{kh_s}{b} + h_s h_a \ell n\left(\frac{b}{a}\right)}\left(\frac{k}{h_a b} + \ell n(b)\right)$$

The temperature distribution in the pipe wall T(r) may be obtained by substituting the constants c_1 and c_2 in the above expressions into the solution in Equation (c).

9.5 Solution of Partial Differential Equations for Transient Heat Conduction Analysis (p.298)

The partial differential equation presented below and also in in Equation (9.15) is valid for the general case of heat conduction in solids includes transient cases in which the induced temperature field T(**r**,t) varies with time t.

$$\nabla^2 T(\mathbf{r},t) + \frac{Q(\mathbf{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t} \tag{9.15}$$

where **r** = the position vector and t = time. Q(r,t)= the heat generation by the material in unit volume and time, k, α = thermal conductivity and thermal diffusivity of the material respectively, with k to be a measure of how well material can conduct heat and the latter α is a measure of how fast the material can conduct heat.

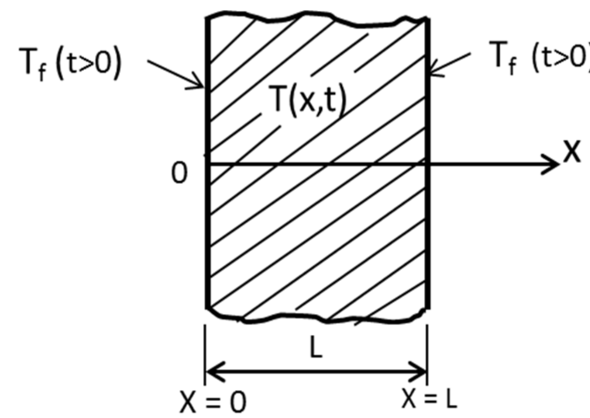
The position vector **r** may be in rectangular coordinates: (x,y,z) or in cylindrical polar coordinate system (r,θ,z).

The complexity in transient heat conduction analysis is that not only we need to specify the position (r) where the temperature of the solid is accounted for, but we will also need to specify the time t at which the temperature of the solid occurs. We thus need to specify both theboundary and initial conditions such as described in Section 9.4.6 for complete solution of the temperature filed in the solid.

In this section, we will demonstrate how the separation of variables technique described in Section 9.3 will be used to solve this type of problems in both rectangular and cylindrical polar coordinate systems.

9.5.1 Transient heat conduction analysis in rectangular coordinate system (p.298)

The case that we will present here involves a large flat slab made of a material with thermal conductivity k.
The slab has a thickness L as illustrated in Figure 9.8. It has an initial temperature distribution that can be described by a specified function of f(x), and the temperatures of both its faces are maintained at temperature T_f at time t > 0.



We need to determine the temperature variation in the slab with time t, i.e. T(x,t) in the figure after the temperature of both faces of the slab are maintained at T_f.

The physical situation of this example is that the flat slab has an initial temperature variation through its thickness fits a function T(x.0) = f(x) –a given temperature distribution. Both its surfaces are maintained at a constant temperature T_f at time $t > 0^+$ for t > 0. One may imagine that the temperature in the slab will continuously varying with time t, until the temperature in the entire slab reaches a uniform temperature T_f. The purpose of our subsequent analysis, however, is to find the transient temperature T(x,t) in the slab before it reaches the ultimate uniform temperature of T_f.

We may also recognize a fact that the geometry of a large flat slab is a good approximation for the situation of a circular cylinder with large diameter with a large ratio of D/d in which D is the nominal diameter of the hollow cylinder and d is the thickness of the wall of the hollow cylinders. The solution obtained from this analysis of flat slab may thus be used for large hollow cylinders such as pressure vessels of large diameters such as for nuclear reactor vessels in nuclear power plants.

9.5.1 Transient heat conduction analysis in rectangular coordinate system –Cont'd (p.299)

The governing differential equation for the aforementioned physical situation may be deduced from heat conduction equations in Equations (9.14a) and (9.15) with the thermal conductivity of the slab material $k_x = k_y = k_z = k$ for being an isotropic material. The term Q(x,y,z,t) in Equation (9.14a) and Q(**r**,t) in Equation (9.15) are deleted because the slab does not generate heat by itself. Consequently, the equation that matches the the present physical situation becomes:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \qquad (9.18)$$

With the initial condition (IC):

$$T(x,t)\big|_{t=0} = T(x,0) = f(x) \qquad (9.19a)$$

and the following boundary conditions (BC):

$$T(x,t)\big|_{x=0} = T(0,t) = T_f \qquad t>0 \qquad (9.19b)$$

$$T(x,t)\big|_{x=L} = T(L,t) = T_f \qquad t>0 \qquad (9.19c)$$

We may solve the partial differential equation in Equation (9.18) by using Laplace transform method described in Section 6.5.2 (p. 180) or 9.3 (p.287) by transforming the variable "t" to parametric domain , or use the separation of variables technique as described in Section 9.3.1. However, we may circumvent our effort in the solution of Equation (9.18) by using the separation of variables method with converting the non-homogeneous BCs in Equation (9.19b,c) to homogeneous BCs by the following substitution of u(x,t) to T(x,t):

$$u(x,t) = T(x,t) - T_f \qquad (9.20)$$

The above relation in Equation (9.20) will result in the revised PDEs in Equation (9.18) into the following form:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} \qquad (9.21)$$

with the revised initial condition:

$$u(x,t)\big|_{t=0} = u(x,0) = f(x) - T_f \qquad (a)$$

and the 2 converted boundary conditions:

$$u(x,t)\big|_{x=0} = u(o,t) = T(x,t)\big|_{x=0} - T_f = T_f - T_f = 0 \qquad (b)$$

$$u(x,t)\big|_{x=L} = u(L,t) = T(x,t)\big|_{x=L} - T_f = T_f - T_f = 0 \qquad (c)$$

We are now ready to solve the equation in (9.21) and the associate initial and boundary conditions in Equations (a,b,c) using the separation of variables method as presented below:

We will proceed by letting:

$$u(x,t) = X(x)\tau(t) \qquad (9.22)$$

Substituting the relationship in Equation (9.22) into Equation (9.21) will lead to the following expressions:

$\frac{\partial^2[X(x)\tau(t)]}{\partial x^2} = \frac{1}{\alpha}\frac{\partial[X(x)\tau(t)]}{\partial t}$ leads to: $\tau(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha}X(x)\frac{\partial \tau(t)}{\partial t}$ and this equality can now be expressed in ordinary derivatives. $\tau(t)\frac{d^2 X(x)}{dx^2} = \frac{1}{\alpha}X(x)\frac{d\tau(t)}{dt}$ in which the partial derivatives on either sides are obtained.

9.5.1 Transient heat conduction analysis in rectangular coordinate system –Cont'd

The expression that we just derived, as shown below

$$\tau(t)\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}X(x)\frac{d\tau(t)}{dt}$$

can be expressed in a slight different form after re-arranging the terms to another equality:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt}$$

The above expression shows a very interesting but unique feature:

The LHS of the above expression involves the variable x only =
The RHS of the same expression involves the variable t only

The **ONLY** condition such an equality can exist is to have **both sides of the expression to equal a CONSTANT**!! (we may prove that the constant must be a NEGATIVE constant).

Consequently, we may have the following valid equality:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt} = -\beta^2 \qquad (9.23)$$

where β is the "separation constant" and it can be either positive or negative constant.
Equation (9.23) results in the following 2 separate ordinary differential equations:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt} = -\beta^2$$

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \qquad (9.24)$$

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \qquad (9.25)$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{\alpha}\frac{1}{\tau(t)}\frac{d\tau(t)}{dt} = -\beta^2 \rightarrow$$

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \qquad (9.24)$$

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \qquad (9.25)$$

The solution X(x) and τ(t) in respective Equations (9.24) and (9.25) requires the specific conditions for both these equations. Equation (9.22) is used in conjunction with those given initial and boundary conditions in Equations (a,b,c) will get us the following required equivalent conditions:

X(0)= 0 and X(L) = 0 (e1, e2)

for Equation (9.24).

Solution X(x) in Equation (9.24) is readily found from Section 8.2 with the form:

X(x) = A cosβx + B sinβx (f)

The arbitrary constant A in Equation (f) can be determined by Equation (e1) to be zero, which leaves Bsinβx=0. the use of the given condition in Equation (e2) leads to BsinβL=0, which leads to either B=0 or sinβL=0; Since B≠0 (to avoid a non-trivial solution of X(x)=0), the only choice for us is to let **sinβL = 0** (9.26)

We will quickly realize that there are multiple values of the separation constant β that satisfy Equation (9.26). These are: β = nπ, with n = 1,2,3,……….Alternatively, we may express the separation constant β in the following form: $\beta_n = \frac{n\pi}{L} \qquad (n = 1,2,3,\ldots\ldots\ldots\ldots\ldots)$ (9.27)

Consequently, the function X(x) in Equation (9.24) takes the form:

$$X(x) = B_1\sin\frac{\pi x}{L} + B_2\sin\frac{2\pi x}{L} + B_3\sin\frac{3\pi x}{L} + \ldots\ldots\ldots\ldots$$
$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad (n = 1, 2, 3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots) \qquad (9.28)$$

We are now ready to solve the other function $\tau(t) in\ Equation\ (9.25) by\ the\ follwoing\ steps$

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \qquad (9.25)$$

Solution of this first order differential equation is:

$$\tau(t) = C_n e^{-\alpha\beta_n^2 t} \qquad (9.29)$$

where C_n with n = 1, 2, 3,……are multi-valued integration constants corresponding to the multivalued β_n in the solution.

The general solution of Equation (9.21) can thus be obtained by substituting the solutions X(x) in Equation (9.28) and τ(t) in Equation (9.29) into Equation (9.22) to give:

$$u(x,t) = \sum_{n=1}^{\infty} C_n B_n e^{-\alpha\beta_b t} \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n\, e^{-\alpha\beta_n t} \sin\frac{n\pi x}{L} \qquad (9.30)$$

The multi-valued constant coefficients $b_n=C_nB_n$ in Equation (9.30) may be determined by the last available initial condition in Equation (a) in which u(x,o) = f(x)-T_f. Consequently, we have:

$$u(x,0) = f(x) - T_f = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (9.31)$$

where f(x) and T_f are the given initial temperature distribution in the slab and the contacting bulk fluid temperature respectively.

9.5.1 Transient heat conduction analysis in rectangular coordinate system –Cont'd

Determination of the multi-valued constant coefficients **b_n** in Equation (9.30) on P.302:

We will use the "orthogonality property of integrals of trigonometry functions" for the above task. The two applicable properties are presented below:

$$\int_0^p \sin\frac{n\pi x}{p}\sin\frac{m\pi x}{p}dx = \begin{cases} 0 & if \;\; m \neq n \\ p/2 & if \;\; m = n \end{cases} \tag{9.32}$$

Following steps are taken in determining the coefficient b_n with n = 1,2,3,…., in Equation (9.31):

Step 1: Multiply both side of Equation (9.26) with function $\sin\frac{n\pi x}{L}$

$$\left(\sin\frac{n\pi x}{L}\right)\left[f(x)-T_f\right] = \left(\sin\frac{n\pi x}{L}\right)\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L} \tag{g}$$

Step 2: Integrate both sides of Equation (g) with integration limits of (0,L):

$$\int_0^L\left(\sin\frac{n\pi x}{L}\right)\left[f(x)-T_f\right]dx = \int_0^L \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}dx = \sum_{n=1}^{\infty}\int_0^L b_n\left(\sin\frac{n\pi x}{L}\right)^2 dx \tag{h}$$

Step 3: Make use of the orthogonality of the harmonious functions like sine and cosine with the relationships in Equation (9.32): $\int_0^L\left(\sin\frac{n\pi x}{L}\right)\left[f(x)-T_f\right]dx = b_n\left(\frac{L}{2}\right)$ leading to:

$$b_n = \frac{2}{L}\int_0^L\left[f(x)-T_f\right]\sin\frac{n\pi x}{L}dx \tag{9.33}$$

We thus have the solution of Equation (9.21) to be: $u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L\left[f(x)-T_f\right]\sin\frac{n\pi x}{L}dx\right\}e^{-\alpha\beta_n^2 t}\sin\frac{n\pi x}{L}$

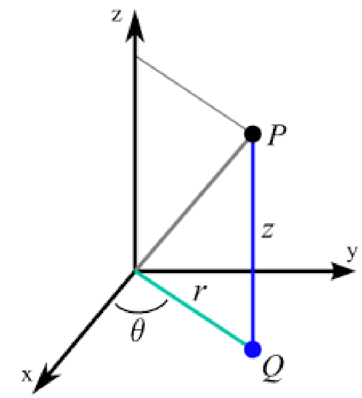
The solution of T(x,t) in Equation (9.18) for the temperature distribution in the slab can thus be obtained by the relationship expressed in Equation (9.20) to take the form:

$$T(x,t) = T_f + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L\left[f(x)-T_f\right]\sin\frac{n\pi x}{L}dx\right\}e^{-\alpha\beta_n^2 t}\sin\frac{n\pi x}{L} \tag{9.34}$$

It will not be hard for us to envisage that $T(x,\infty)\rightarrow T_f$ in Equation (9.34) – a solution in reality.

9.5.2 Transient heat conduction analysis in cylindrical polar coordinate system (p.303)

There are many mechanical engineering equipment having geometry that can be better defined by cylindrical polar coordinates (r,θ,z) such as illustrated in the figure to the right:



Cylinders, pipes, wheels, disks, etc. all fit to this kind of geometry such asshown in Figure 9.9..

It is desirable to know how to handle heat conduction in solids of these geometry.

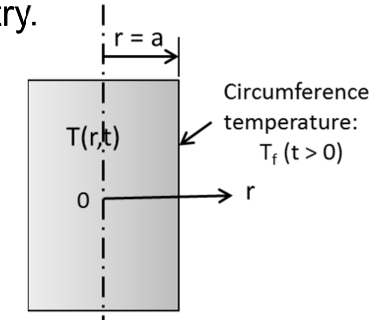


Figure 9.9

We will present the case of solving heat conduction problem using the separation variable technique in a solid cylinder with radius a as shown in Figure 9.9.

The cylinder is initially with a given temperature distribution of f(r). It is submerged in a fluid with bulk fluid temperature T_f.at time $t+0^+$.

The situation in real application is like having a hot round solid cylinder initially with a temperature variation from hot center cooling down towards its circumference surface described by function f(r). It is a classical case of “quenching” operation in a metal forming operation.

The surrounding contacting liquid at a cooler temperature T_f is vigorously agitated so that the heat transfer coefficient h of the fluid at the contact surface may be treated as “∞” in Equation (9.17d) on p, 295, leading to the boundary temperature of the solid cylinder to be T_f, as stated in the problem. The temperature field in the solid cylinder may be represented by the function T(r,t), in which r = radial coordinate and t is the time into the heat conduction in the solid.

9.5.2 **Transient heat conduction analysis in cylindrical polar coordinate system –** Cont'd (p.303)

The applicable PDE for the current application may be deduced from Equation (9.14b) by dropping the second and other terms in the right-hand-side of that equation, resulting in:

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r} \qquad (9.35)$$

where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity of the cylinder material with ρ and c being the mass density and specific heat of the cylinder material respectively.

We will have the given initial condition: $T(r,t)\big|_{t=0} = T(r,0) = f(r)$ (a)

and boundary conditions: $T(r,t)\big|_{r=a} = T(a,t) = T_f \qquad t>0$ (b1)

The other "inexplicit" boundary condition for solid cylinders or disks is that the temperature at the center of the cylinder or disk must be a finite value at all times. Conversely this implicit boundary condition for the current case meant to be:

$$T(r,t)\big|_{r=0} = T(0,t) \neq \infty \quad or \quad T(0,t) = finite\ value \qquad (b2)$$

with the PDE in (9.35) and the initial and boundary conditions specified in Equations (a), (b1) and (b2) as specified above, we may proceed to solve for the transient temperature distribution T(r,t) in Equation (9.35) by using the separation of variables technique similar to what we did in the proceeding Section 9.5.1.

Again, for the same reason as in the previous case, we will first convert the non-homogeneous boundary condition in Equation (b1) to the form of homogeneous condition by letting:

$$u(r.t) = T(r,t) - T_f \qquad (c)$$

Accordingly, Equation (9.35) and the original initial and boundary conditions will have the forms:

$$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \qquad (9.36)$$

with $\quad u(r,t)\big|_{t=0} = u(r,0) = f(r) - T_f \qquad for \quad t=0$ (d)

and $\quad u(r,t)\big|_{r=a} = u(a,t) = 0 \qquad for\ \ t>0$ (e)

We thus have the PDE: $$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t}=\frac{\partial^2 u(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,t)}{\partial r} \qquad (9.36)$$

with conditions:

$$u(r,t)\big|_{t=0}=u(r,0)=f(r)-T_f \qquad for \quad t=0 \qquad (d)$$

$$u(r,t)\big|_{r=a}=u(a,t)=0 \qquad for \ t>0 \qquad (e)$$

We realize that there are two independent variables in the function u(r,t) in Equation (9.36), so, we will use the following formula to separate the two variables in the function u(r,t) by letting:

$$U(r,t) = R(r)\tau(t) \qquad (9.37)$$

Upon substituting the above relation in Equation (9.37) into Equation (9.36) will result in the following Expressions::

$$\frac{1}{\alpha}\frac{\partial[R(r)\tau(t)]}{\partial t}=\frac{\partial^2[R(r)\tau(t)]}{\partial r^2}+\frac{1}{r}\frac{\partial[R(r)\tau(t)]}{\partial r} \quad or$$

$$\frac{R(r)}{\alpha}\frac{\partial\tau(t)}{\partial t}=\tau(t)\frac{\partial^2 R(r)}{\partial r^2}+\frac{\tau(t)}{r}\frac{\partial R(r)}{\partial r} \qquad (f)$$

Equation (f) offers the legitimacy of converting the partial derivatives of R(r) and τ(t) to ordinary derivatives as shown below:

$$\frac{1}{\alpha\tau(t)}\frac{d\tau(t)}{dt}=\frac{1}{R(r)}\left[\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}\right] \qquad (g)$$

We notice that the LHS of Equation (g) involves variable t only whereas the RHS of the same expression involve the other variable r only. The only way that such equality can exist is for both sides in Equation (g) to be equal to a same negative separation constant β. We thus have the following relationship:

$$\frac{1}{\alpha\tau(t)}\frac{d\tau(t)}{dt}=\frac{1}{R(r)}\left[\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}\right]=-\beta^2 \qquad (9.38)$$

9.5.2 Transient heat conduction analysis in cylindrical polar coordinate system – Cont'd

Solution of partial differential equation: $$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \tag{9.36}$$

with conditions: $$u(r,t)\big|_{t=0} = u(r,0) = f(r) - T_f \qquad for \quad t=0 \tag{d}$$

$$u(r,t)\big|_{r=a} = u(a,t) = 0 \qquad for \ t>0 \tag{e}$$

We can thus split Equation (9.38) into the following two separate ordinary differential equations:

$$\frac{1}{\alpha\tau(t)}\frac{d\tau(t)}{dt} = \frac{1}{R(r)}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr}\right] = -\beta^2$$

$$\frac{d\tau(t)}{dt} + \alpha\beta^2\tau(t) = 0 \tag{9.39}$$

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \beta^2 r^2 R(r) = 0 \tag{9.40}$$

The solution of Equation (9.39) is identical to Equation (9.29) in the form:

$$\tau(t) = c_n e^{-\alpha\beta_b^2 t} \tag{h}$$

where the constant coefficients c_n with n = 1, 2, 3,….. is a multivalued integration constants.

We notice that Equation (9.40) is special case of the Bessel equation in Equation (2.27) on p.56 with order n = 0. Consequently, the solution of Equation (9.40) can be expressed by the Bessel functions given in Equation (2.28) on the same page with n = 0 in the following form:

$$R(r) = A\,J_0(\beta r) + B\,Y_0(\beta r) \tag{9.41}$$

where the constant coefficients A and B will be determined by the boundary conditions stipulated in Equations (d) and (e).

Solution of partial differential equation: $$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t}=\frac{\partial^2 u(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,t)}{\partial r} \qquad (9.36)$$

with conditions: $$u(r,t)\big|_{t=0}=u(r,0)=f(r)-T_f \qquad for \quad t=0 \qquad (d)$$

$$u(r,t)\big|_{r=a}=u(a,t)=0 \qquad for \;\; t>0 \qquad (e)$$

We have solve the differential equation in Equation (9.40) to be the expression given in Equation (9.41):

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}+\beta^2 r^2 R(r)=0 \qquad (9.40)$$

$$R(r) = A\, J_0(\beta r) + B\, Y_0(\beta r) \qquad (9.41)$$

where A and B are two arbitrary constants to be determined by the two boundary conditions applicable in this case are: R(0) for u(0,t) and thus T(0,t). For the condition R(0), we will have, from Equation (h) in the form: $R(0)=AJ_0(0)+BY_o(0)$, we realize that $J_0(0) = 1.0$ from Figure 2.45 (p.56), but $Y_o(0)\rightarrow-\infty$ as indicated in the same figure. The latter indicates that R(0), therefore $T(0) \rightarrow -\infty$ (an unbounded temperature at the center of the solid cylinder, which is obviously not a realistic solution. The only way that we may avoid this unrealistic situation is to let the constant B = 0.

Consequently, we have the solution in Equation (h) to take the form: $R(r) = A\, J_0(\beta r)$ (j)

The boundary condition in Equation (e) will lead to the expression: $R(a) =A\, J_0(\beta a) = 0$, which requires either: A = 0, or $J_0(\beta a) = 0$. Since the coefficient B in Equation (h) is already set to be zero (0), to let A=0 will mean the function R(r)=0, an unacceptable trivial solution for the temperature T(r,t). We are thus left with the only option to have:

$$J_0(\beta a) = 0 \qquad (9.42)$$

Equation (9.42) offers the values of the separation constant β in Equation (9.38) because $J_0(x) = 0$ is an equation that has multiple roots (see Figure 2.45(a) on p.56 like sin(βL) = 0 in Equation (9.26) on p.301. The roots of the equation $J_0(\beta a) = 0$ in Equation (9.42) may be found either from the Figure 2.45(a) on p.56, or from math handbooks.

Solution of partial differential equation: $$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \qquad (9.36)$$

with conditions:
$$u(r,t)\big|_{t=0} = u(r,0) = f(r) - T_f \qquad for \quad t=0 \qquad (d)$$
$$u(r,t)\big|_{r=a} = u(a,t) = 0 \qquad for \ t>0 \qquad (e)$$

We have set the solution u(x,t) of Eqution (9.36) in the form of u(r,t) = R(r)$\tau(t)$ *in Equation* (9.17, *and we* have solved $R(r) = A\,J_0(\beta r)$ in Equation (j), and $\tau(t) = c_n e^{-\alpha\beta_b^2 t}$ in Equation (h). We thus have the solution u(r,t) in the following form :

$$u(r,t) = AC_n e^{-\alpha\beta_n^2 t} J_0(\beta r)$$

Since both A and C_n are constants, and the latter C_n is a multivalued constants with n = 1,2,3,…., we may express the complete solution u(r,t) in the form:

$$u(r,t) = \sum_{n=1}^{\infty} b_n\, e^{-\alpha\beta_n^2 t} J_0(\beta_n r) \qquad (9.44)$$

where the multi-valued constant b_n may be determined by the conditions in Equations (d) and (e).

We thus have the following expression after apply the initial condition in Equation (d):

$$u(r.0) = f(r) - T_f = \sum_{n=1}^{\infty} b_n\, J_0(\beta_n r) = b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + ... \qquad (9.45)$$

where f(r)-T_f in Equation (d) are given conditions with the PDE in Equation (9.35), and b_n in Equation (9,45) may be determined by following a similar procedure as outlined in Section 9.5.1 using the "orthogonality properties" of trigonometric functions in Equation (9.32) on p.302. However, we will use the Fourier-Bessel relation in determining the coefficients b_n in Equation (9.45) in the present case.

Solution of partial differential equation: $\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r}$ (9.36)

with conditions:

$$u(r,t)\big|_{t=0} = u(r,0) = f(r) - T_f \qquad for \quad t=0 \qquad \text{(d)}$$

$$u(r,t)\big|_{r=a} = u(a,t) = 0 \qquad for \ t>0 \qquad \text{(e)}$$

The **Fourier-Bessel relation** has the form (p.307):

$$\int_0^a rJ_0(\beta_m r)J_0(\beta_n r)dr = \begin{cases} 0 \quad if\ \beta_m \neq \beta_n & \text{for different arguments in the Bessel functions in the integral} \\ \int_0^a r[J_0(\beta_n r)]^2 dr \ \ if \ \ \beta_m = \beta_n & \text{for same arguments in the Bessel functions in the integral} \end{cases}$$

We will multiply both sides of Equation (9.45) by the following series of Bessel functions: $[rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + ...]$ as shown in the following expression:

$$[rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + ...][f(r) - T_f]$$
$$= [rJ_0(\beta_1 r) + rJ_0(\beta_2 r) + rJ_0(\beta_3 r) + ...][b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + ...]$$

and the expansion of both sides of the above expression will result in:

$$rJ_0(\beta_n r)[f(r) - T_f] = rJ_0(\beta_n r)[b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + ...] \qquad \text{(k)}$$

Integrating both sides of Equation (k) with respect to variable r will result in:

$$\int_0^a rJ_0(\beta_n r)[f(r) - T_f]dr = \int_0^a rJ_0(\beta_n r)[b_1 J_0(\beta_1 r) + b_2 J_0(\beta_2 r) + b_3 J_0(\beta_3 r) + ...]dr \quad for\ n=1,2,3,........$$

$$= \int_0^a rb_n[J_0(\beta_n r)]^2 dr + \int_0^a rb_1[J_0(\beta_1 r)J_0(\beta_2 r) + J_0(\beta_1 r)J_0(\beta_3 r) + ...]dr + ... \qquad (\ell)$$

The Fourier-Bessel relation enables us to eliminate the 2nd part of the Bessel functions, and result in:

$$\int_0^a rJ_0(\beta_n r)[f(r) - T_f]dr = b_n \int_0^a r[J_0(\beta_n r)]^2 dr \qquad \text{(m)}$$

We may thus obtain the multi-valued coefficient b_n to be: $b_n = \frac{2}{a^2 J_1^2(\beta_n a)}\int_0^a r[f(r) - T_f]J_0(\beta_n r)dr$ (9.46)

9.5.2 Transient heat conduction analysis in cylindrical polar coordinate system – End

Solution of partial differential equation: $$\frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \quad (9.36)$$

with conditions:

$$u(r,t)\big|_{t=0} = u(r,0) = f(r) - T_f \qquad for \quad t=0 \quad (d)$$

$$u(r,t)\big|_{r=a} = u(a,t) = 0 \qquad for \;\; t>0 \quad (e)$$

The solution of u(r,t) in Equation (9.36) thus has the form:

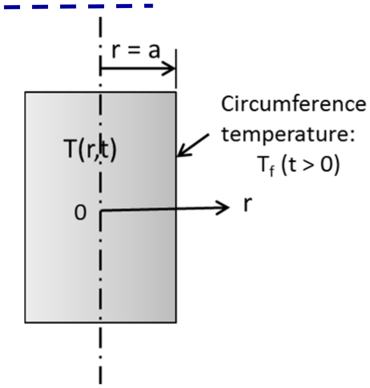
$$u(r,t) = \sum_{n=1}^{\infty} b_n\, e^{-\alpha\beta_n^2 t} J_0(\beta_n r) \quad (9.44)$$

where the coefficients b_n are obtained fro the integral in Equation (9.46):

$$b_n = \frac{2}{a^2 J_1^2(\beta_n a)} \int_0^a r\left[f(r) - T_f\right] J_0(\beta_n r)\,dr \quad (9.46)$$

We may obtain the transient temperature distribution in the cylinder T(r,t) by the relation derived from Equation (c) as: $T(r,t) = T_i + u(r,t)$.

We thus have the solution of the temperature distribution in the cylinder to be:



$$T(r,t) = T_f + \sum_{n=1}^{\infty} b_n\, e^{-\alpha\beta_n^2 t} J_0(\beta_n r) \quad \mathbf{(9.47)}$$

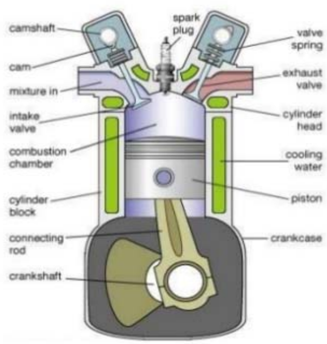
where the multi-valued coefficients are computed from Equation (9.46)

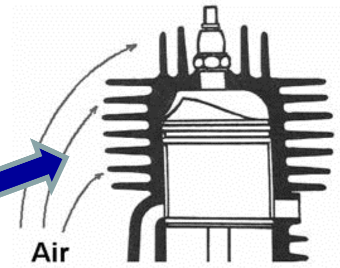
We notice the appearances of Bessel functions in the solution of this problem. It is <u>normal to see such appearances of Bessel functions in solid geometry involving circular geometry, such as cylinders, disks, and even solids of spherical geometry</u>.

9.6 **Solution of Partial Differential Equations for Steady-State Heat Conduction Analysis**

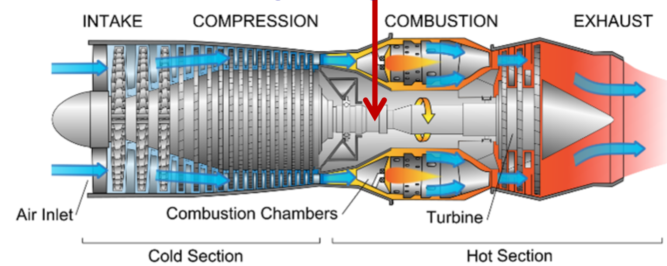
Often, we are required to find the temperature distributions in solid machine structures with stable heat flow patterns, which makes the temperature distributions in the solids independent of time variation, i.e., the steady state heat conduction. Following are examples on heat flow in the machines in steady-state conditions:

Internal Combustion Engines

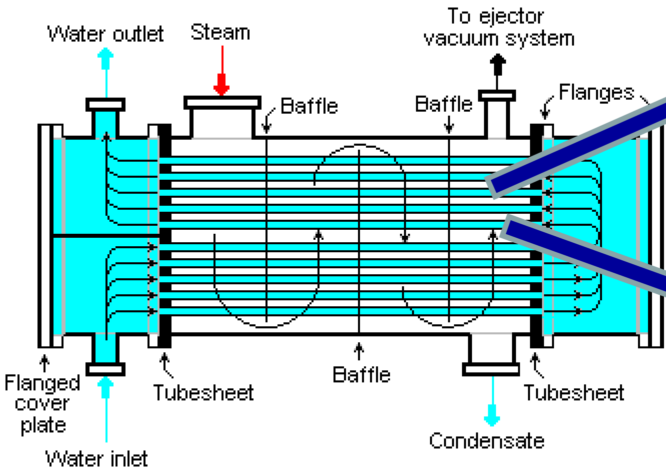




Jet engine-gas turbine

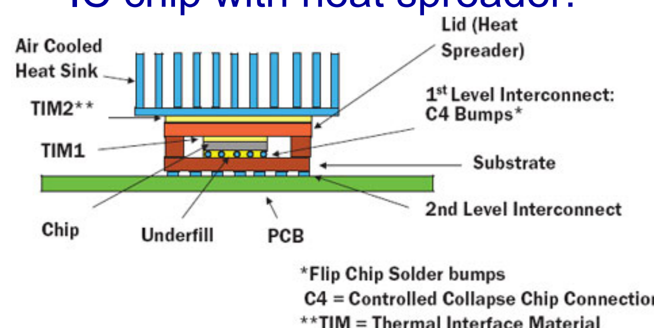


Tubular heat exchanger:



Tubes with fins

IC chip with heat spreader:



9.6 Partial Differential Equations for Steady-State Heat Conduction Analysis (p.308)

Mathematical representation of multi-dimensional heat conduction in solids is available by using the partial differential equation without the term related to time variable t. The PDE in Equation (9.15) on p.293 is reduced to the following form

$$\nabla^2 T(\mathbf{r}) + \frac{Q(\mathbf{r})}{k} = 0 \tag{9.48}$$

where the position vector **r** represents (x,y,z) in rectangular coordinate system, or (r,θ,z) in cylindrical polar coordinate system.

Equation (9.48) is further reduced to the "Laplace equations" in the following form if no heat is generated by the solid:

$$\nabla^2 T(\mathbf{r}) = 0 \tag{9.49}$$

We will demonstrate the solution of PDEs for steady-state heat conductions in multi-dimensional solid structure components using separation variables technique in both rectangular and cylindrical polar coordinate systems in the subsequent presentations.

9.6.1 Steady-State Heat Conduction Analysis in Rectangular Coordinate System (p.308)

We will demonstrate the use of the Laplace equation in Equation (9.49) for the temperature distribution in a square plate with temperature in its three edges maintained at constant temperatures at 0°C and the other edge at 100°C, as illustrated in Figure 9.10.

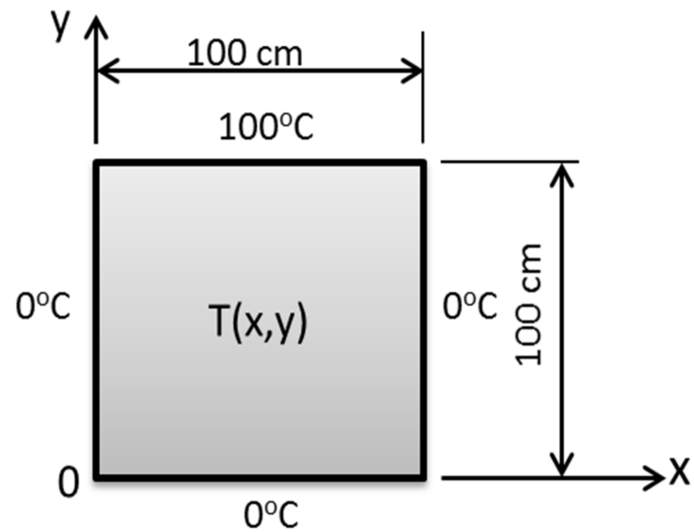


Figure 9.10

Solid plate structure components are common in the heat spreaders in internal combustion engines, tubular heat exchanges, and heat spreaders for microchips, as illustrated in the last slide.

In the present case, heat flows from the heat source at the top face with higher temperature towards the heat sinks in the other edges at lower temperatures. There is no heat flows across the thickness of the plate with an assumption that both the plan faces of the plate is thermally insulated. The induced temperature field thus covers over the plane area of the plate, and the temperature distribution in the plane of the plate is represented by the function **T(x,y).**

We have the applicable PDE expressed in Equation (9.50), and specified boundary conditions in Equations (a1), (a2), (a3) and (a4).

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \qquad (9.50)$$

with $0 \le x \le 100 \;\; and \;\; 0 \le y \le 100$

$$T(x,y)\big|_{x=0} = T(0,y) = 0 \qquad (a1)$$

$$T(x,y)\big|_{x=100} = T(100,y) = 0 \qquad (a2)$$

$$T(x,y)\big|_{y=0} = T(x,0) = 0 \qquad (a3)$$

$$T(x,y)\big|_{y=100} = T(x,100) = 100 \qquad (a4)$$

Solution of Partial Differential Equation using Separation of Variables Method (p.309):

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \qquad (9.50)$$

$$0 \le x \le 100 \quad and \quad 0 \le y \le 100$$

Boundary conditions:

$$T(x,y)\big|_{x=0} = T(0,y) = 0 \qquad (a1)$$

$$T(x,y)\big|_{x=100} = T(100,y) = 0 \qquad (a2)$$

$$T(x,y)\big|_{y=0} = T(x,0) = 0 \qquad (a3)$$

$$T(x,y)\big|_{y=100} = T(x,100) = 100 \qquad (a4)$$

There are two variables x and y in the solution of temperature function T(x,y), we will thus let:

$$\mathbf{T(x,y) = X(x)\,Y(y)} \qquad (b)$$

in which function X(x) involves only variable x, and function Y(y) involves variable y only.

Substitute Equation (b) into Equation (9.50), and after re-arranging terms, yields the following expression:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

We will use the same argument as we did in Sections 9.5.1 and 9.5.2 that the only way the above equality can exist is to have both sides of the equality to be equal to a negative separation constant.

We will thus have the following equality:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta^2 \qquad (c)$$

9.6.1 Steady-State Heat Conduction Analysis in Rectangular Coordinate System – Cont'd

Solution of PDE in Equation (9.50) using Separation of Variables Method-Cont'd:

Equation (c) leads to the split of the PDE in Equation (9.50) into two ordinary differential equations (ODEs) after the separation of the variables x and y as shown below:

The original PDE in Equation (9.50):

$$\frac{\partial^2 T(x,y)}{\partial x^2}+\frac{\partial^2 T(x,y)}{\partial y^2}=0$$

$$\frac{d^2X(x)}{dx^2}+\beta^2X(x)=0 \qquad \text{(d)}$$

$$\mathrm{X}(0)=0 \qquad \text{(f1)}$$

$$\mathrm{X}(100)=0 \qquad \text{(f2)}$$

$$\frac{d^2Y(y)}{dy^2}-\beta^2Y(y)=0 \qquad \text{(e)}$$

$$\mathrm{Y}(0)=0 \qquad \text{(f3)}$$

Both Equations (d) and (e) are homogeneous 2nd order differential equations with the solutions methods available in Section 8.2. We will shown the solutions of these two equations in the following forms:

Solution of Equation (d): $$X(x)=A\cos\beta x+B\sin\beta x \qquad \text{(g)}$$

Solution of Equation (e): $$\mathrm{Y(y) = C\ cosh\beta y + D\ sinh\beta y} \qquad \text{(k)}$$

We may obtain the expression for the multi-valued separation constant β to be the solution of the characteristic equation of sin(100β) = 0, and the constant coefficient A = 0 upon substituting The boundary conditions in Equations (f1) and (f2) into Equation (g). We have thus obtained the multi-valued separation constants β_n = (nπ)/L with L=100 and n = 1,2,3,….,n from the roots of the equation sin(100β) = 0. We can thus express the function as:

$$X(x) = B_n \sin\beta_n x \qquad \text{(j)}$$

in which B_n with n = 1,2,3,….,n are the multi-valued constant coefficients to be determined later.

Solution of Partial Differential Equation (9.50) using Separation of Variables Method-Cont'd:

The original partial differential Equation (9.50):

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

$$\frac{d^2 X(x)}{dx^2} + \beta^2 X(x) = 0 \qquad \text{(d)}$$

$$X(0) = 0 \qquad \text{(f1)}$$

$$X(100) = 0 \qquad \text{(f2)}$$

$$\frac{d^2 Y(y)}{dy^2} - \beta^2 Y(y) = 0 \qquad \text{(e)}$$

$$Y(0) = 0 \qquad \text{(f3)}$$

Next, we will solve Equation (e), with a solution (p.310):

$$Y(y) = C \cosh\beta y + D \sinh\beta y \qquad \text{(k)}$$

The boundary condition in Equation (f3) would make the constant coefficient C = 0. Consequently, we will have the function Y(y) to take the form:

$$Y(y) = D_n \sinh\beta_n y \qquad \text{with } n=1,2,3,\ldots n \qquad \text{(m)}$$

We have obtain the solution T(x,y) of Equation (9.50) after substituting the expressions of X(x) in Equation (j) and Y(y) in Equation (m) into Equation (b) and result in:

$$T(x,y) = \sum_{n=1}^{\infty} X(x)Y(y) = \sum_{n=1}^{\infty} B_n D_n \left(\sin\frac{n\pi}{100}x\right)\left(\sinh\frac{n\pi}{100}y\right)$$

$$= \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi}{100}x\right)\left(\sinh\frac{n\pi}{100}y\right) \qquad \text{(n)}$$

Solution of Partial Differential Equation (9.50) using Separation of Variables Method-Cont'd:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \qquad (9.50)$$

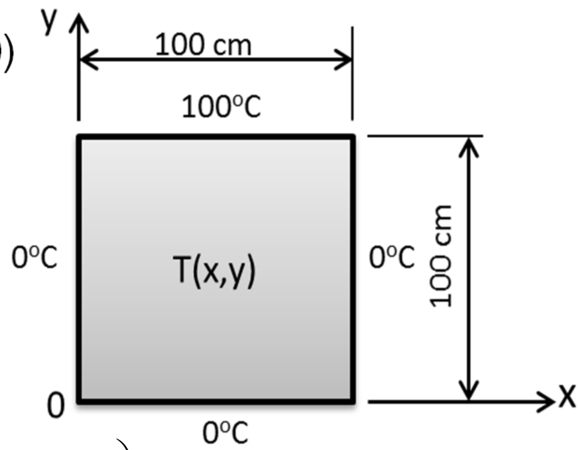
$$0 \le x \le 100 \quad and \quad 0 \le y \le 100$$

$$T(x,y)\big|_{x=0} = T(0,y) = 0 \qquad (a1)$$

$$T(x,y)\big|_{x=100} = T(100,y) = 0 \qquad (a2)$$

$$T(x,y)\big|_{y=0} = T(x,0) = 0 \qquad (a3)$$

$$T(x,y)\big|_{y=100} = T(x,100) = 100 \qquad (a4)$$



$$T(x,y) = \sum_{n=1}^{\infty} X(x)Y(y) = \sum_{n=1}^{\infty} B_n D_n \left(\sin\frac{n\pi}{100}x\right)\left(\sinh\frac{n\pi}{100}y\right)$$

$$= \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi}{100}x\right)\left(\sinh\frac{n\pi}{100}y\right) \qquad (n)$$

The unknown coefficients b_n in Equation (m) may be determined by using the remaining boundary condition in Equation (a4) that T(x,100) = 100, which leads to:

$$T(x,100) = 100 = \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi}{100}x\right)(\sinh n\pi) \quad \text{or} \quad 100 = \sum_{n=1}^{\infty} (b_n \sinh n\pi)\sin\frac{n\pi}{100}x \qquad (p)$$

By following the same procedure in in using the orthogonality of trigonometric functions in Section 9.5.1, on p.298, We will determine the constants b_n in Equation (p) to be:

$$b_n = -\frac{200(\cos n\pi - 1)}{n\pi \sinh n\pi} \qquad with\ n = 1, 2, 3, \ldots\ldots\ldots \qquad (q)$$

Leading to the solution:

$$T(x,y) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n \sinh n\pi}\sin\frac{n\pi}{100}x \sinh\frac{n\pi}{100}y \qquad (9.51)$$

9.6.2 Steady-State Heat Conduction Analysis in Cylindrical Polar Coordinate System (p.311)

We will explore how the separation of variables technique may be used in steady-state heat conduction analysis in cylindrical polar coordinate system by this case illustration.

The case we have here involves a solid cylinder with radius a and length L with temperature at the circumference and the bottom end maintained at 0^oC and the temperature at the top surface is subjected to a temperature distribution that fits a specified function F(r) as shown in Figure 9.11.

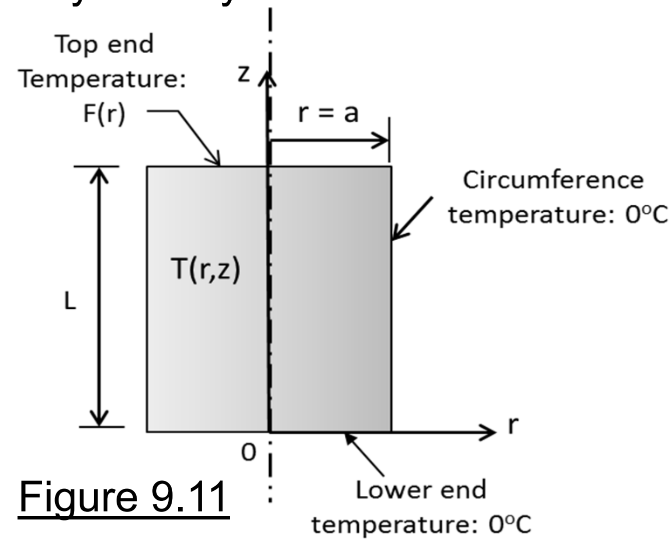


Figure 9.11

We realize the physical situation in which heat flows from the top end of the cylinder in boththe radial and longitudinal direction. We may thus designate the temperature in the cylinder by T(r,z) in a cylindrical polar coordinate system.

The governing PDE for T(r,z) in a steady-state heat conduction as described above may be obtained by selecting the right terms in Equation (9.14b) in cylindrical polar coordinate system in the following form:

$$\frac{\partial^2 T(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,z)}{\partial r}+\frac{\partial^2 T(r,z)}{\partial z^2}=0 \qquad (9.52)$$

with specified boundary conditions: $T(a,z) = 0$ (a1)

and $T(0,z) \neq \infty$ (a2)

Solution of the Partial Differential Equation using Separation of Variable Technique:

$$\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r} + \frac{\partial^2 T(r,z)}{\partial z^2} = 0 \qquad (9.52)$$

with specified boundary conditions: $T(a,z) = 0$ $\quad T(0,z) \neq \infty$ (a1, a2))

$T(r,0) = 0$ (a3)

Following the usual procedures in separation of variable technique (p.312) , we let:

$$T(r,z) = R(r)\,Z(z) \qquad (b)$$

where the functions R(r) and Z(z) involve only one variable r and z respectively.

Upon substituting the above expression in Equation (b) into Equation (9.52), and after re-arranging the terms, we will get the following equality:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} = -\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} \qquad (c)$$

The only way that the above equality can exit is having both sides to be equal to a constant:

We thus have: $$\frac{1}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{1}{rR(r)}\frac{dR(r)}{dr} = -\frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = -\beta^2 \qquad (d)$$

We have thus split the PDE in Equation (9.52) into two separate ODEs as follows:

$$\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r} + \frac{\partial^2 T(r,z)}{\partial z^2} = 0 \;\Rightarrow\; \begin{cases} r\dfrac{d^2 R(r)}{dr^2} + \dfrac{dR(r)}{dr} + r\beta^2 R(r) = 0 & (e) \\[2ex] \dfrac{d^2 Z(z)}{dz^2} - \beta^2 Z(z) = 0 & (f) \end{cases}$$

Satisfying the conditions:

$R(a) = 0$ (g1)

$R(0) \neq \infty$ (g2)

$Z(0) = 0$ (g3)

Solution of the Partial Differential Equation using Separation of Variable Technique-Cont'd:

The solution of the ODE in Equation (e) involves Bessel functions as in the case in Section 9.5.2 and in Equation (9.41) on p.305 to take the form: $R(r) = AJ_0(\beta r) + BY_0(\beta r)$

The condition specified condition in Equation (g2) results in having the constant coefficient in the above expression to be: B = 0, because the second term in the above solution in the above expression cannot be allowed in the expression because $Y_0(0) \to -\infty$, which is not realistic. Hence we have:

$$J_0(\beta_n a) = 0 \qquad (j)$$

The separation constant β is obtained from Equation (j), and there are multiple roots of that equation, with: $\beta = \beta_1, \beta_2, \beta_3, \ldots\ldots \beta_n$ with n = 1,2,3,….,n. The solution of other ODE in Equation (f) is:

$$Z(z) = C\cosh(\beta z) + D\sinh(\beta z) \qquad (k)$$

Substitution of the condition Z(0) = 0 in Equation (a3) into Equation (k) will lead to the constant C = 0. We will thus have: Z(z) = D sinh (βz). However, since Z(z) involve the multi-valued β_n, We may express Z(z) in the form: $Z(z) = D_n \sinh(\beta_n z)$ (m)

We can thus express the solution T(r,z) in Equation (9.52) in the form of:

$T(r,z) = [A_n J_0(\beta_n r)][D_n \sinh(\beta_n z)]$ with n = 1,2,3,…..n, or in a more compact form:

$$T(r,z) = \sum_{n=1}^{\infty} b_n [J_0(\beta_n r)](\sinh \beta_b z) \qquad (9.53)$$

Where b_n are multi-valued constant in the the above equation that may be obtained by using the Fourier-Bessel relation as expressed on P. 307, resulting in the following form:

$$b_n = \frac{2F(r)}{L^2 J_1^2(\beta_n L)\sinh \beta_n L} \int_0^L r J_0(\beta_n r)\,dr \quad with\ n = 1,2,3,\ldots\ldots\ldots\ldots \qquad (n)$$

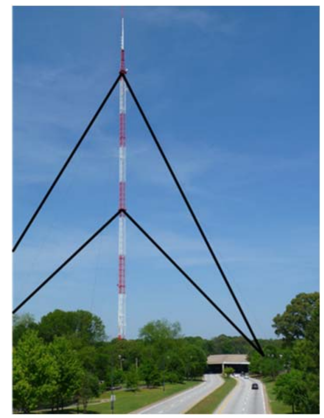
9.7 Partial Differential Equations for Transverse Vibration of Cable Structures (p.314)

Transverse vibration of strings (equivalent to long flexible cable structures in reality) are used commonly used in structures such as power transmission lines, guy wires, suspension bridges.

These structures, flexible in nature, are vulnerable to resonant vibrations, which may result in devastations in public safety and property losses to our society.

Long power transmission lines

Radio tower supported by guy wires



The world famous Golden Gate Suspension Bridge

A cable suspension Bridge at the verge of collapsing:

9.7.1 Derivation of partial differential equation for free vibration of cable structures

We begin our derivation of math models for the vibration analysis of strings (equivalent to long flexible cables) with an initial sagged shape that can be described by a function f(x) as illustrated in Figure 9.15.

Following idealizations (or hypotheses) were made in the derivation of mathematical modes for free vibration analysis of cable structures:

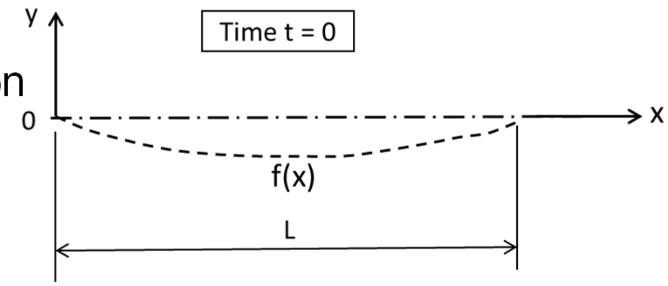


Figure 9.15 A Long Cable Initially in Statically Equilibrium State

(1) The cable is as flexible as a string.
It means that the cable has no strength to resist bending. Hence we will exclude the bending moment and shear forces in our subsequent derivations.

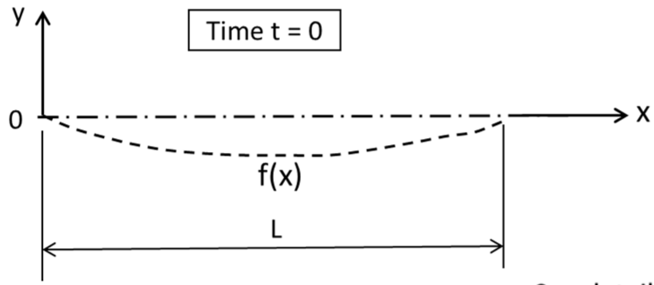
(2) There exists a tension in the string in its free-hung static state as shown in Figure 9.15.
This tension is so large that the weight, but not the mass, of the cable is neglected in the analysis.

(3) Every small segment of the cable along its length, i.e. the segment with a length Δx moves in the vertical direction only during vibration.

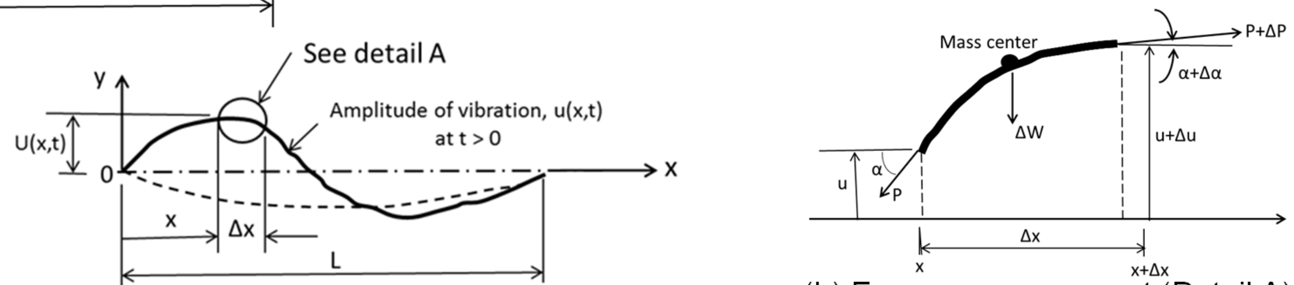
(4) The vertical movement of the cable along the length is small so the slope of the deflection curve of the cable is small.

(5) The mass of the cable along the length is constant, i.e. the cable is made of same material along its entire length.

9.7.1 **Derivation of partial differential equation for free vibration of cable structures** – Cont'd



A slight instantaneous lateral movement of the cable in Figure 9.15 at time t = 0 will result in laterally vibrate up-and-down in the x-y plane as shown in Figure 9.16(a)



(a) Instantaneous shape at time t

(b) Forces on a segment (Detail A)

Figure 9.16 Shape of a vibrating Cable

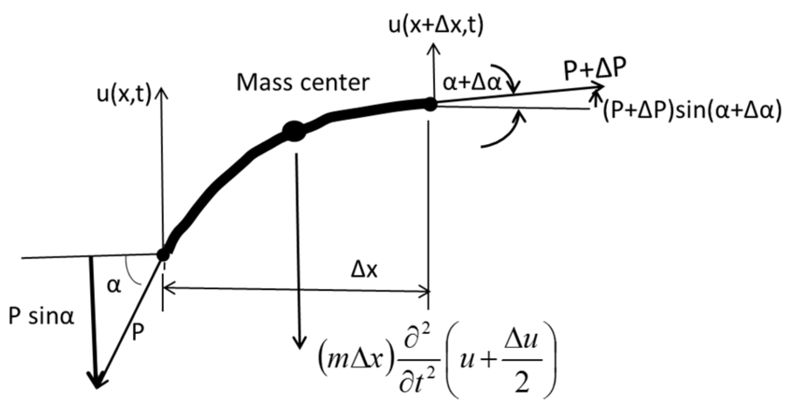


Figure 9.17 Free-body Force diagram of a vibrating cable

Let the mass per unit length of the string be designated by m. The total mass of string in an incremental length Δx in Figure 9.16 (b) and 9.17 will thus be (mΔx).

The condition for a dynamic equilibrium at time t as illustrated in Figure 9.17 according to Newton's second law presented in the *equation of motion* has the following relationship:

Total applied forces: ∑F	=	Mass: m	X	Acceleration: a

9.7.1 Derivation of partial differential equation for free vibration of cable structures – Cont'd

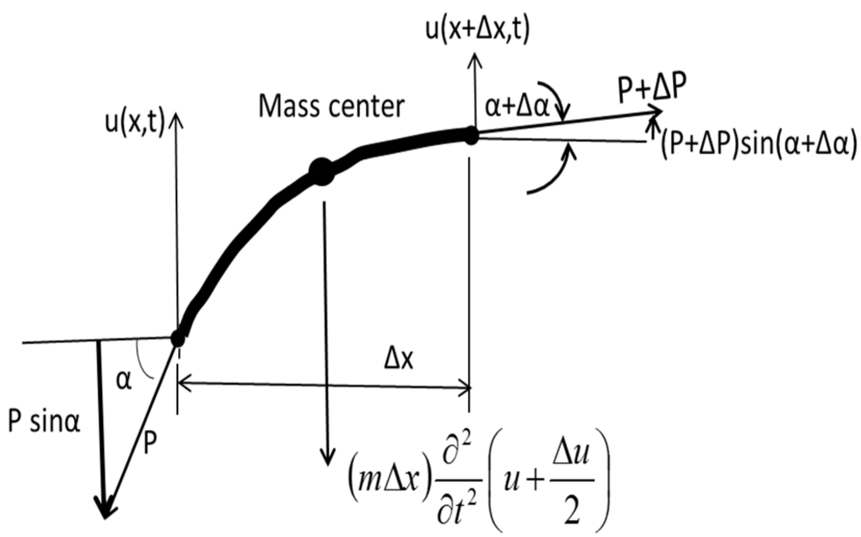


Fig. 9.17 Free-body force diagram

We mentioned in the last slide that the PDF that we will use to model for free lateral vibration analysis of the cable May be derived from Newton's Second law of dynamics:

Total applied forces: ∑F	=	Mass: m	X	Acceleration: a

We further realize that the mass of the segment of the cable in Figure 9.17 may be expressed to be: M= mΔx, in which m= mass of the cable per unit length, and the acceleration is equal to:

$$a = \frac{\partial^2 u(x,t)}{\partial t^2}$$

where u(x,t)= instantaneous deflection (magnitude) of the vibrating cable at x.

We assume this dynamic force acts at the mass center as shown in Figure 9.17.

We may derive the following expression for the dynamic force equilibrium on a small section of the cable at time t:

$$\sum F_y = (P+\Delta P)\sin(\alpha+\Delta\alpha) - P\sin\alpha + (m\Delta x)\frac{\partial^2}{\partial t^2}(u+\frac{\Delta u}{2}) = 0$$

We may delete the term: ΔPsin(α+Δα) in the above expression because both ΔP and Δα are small. We thus have the following for our further derivation:

$$\sum F_y = P\sin(\alpha+\Delta\alpha) - P\sin\alpha + (m\Delta x)\frac{\partial^2}{\partial t^2}(u+\frac{\Delta u}{2}) = 0$$

But since $\sin(\alpha+\Delta\alpha) \approx \tan(\alpha+\Delta\alpha) = \frac{\partial u(x+\Delta x,t)}{\partial x}$, and $\sin\alpha \approx \tan\alpha = \frac{\partial u(x,t)}{\partial x}$,

we will have the following expression after substitutions of the above relationships in the dynamic force equilibrium equation:

$$P\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] = (m\Delta x)\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

If we divide every term in the last expression we will obtain the following expression:

$$P\left[\frac{\partial u(x+\Delta x,t)}{\partial x}-\frac{\partial u(x,t)}{\partial x}\right]=(m\Delta x)\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

or

$$P\,\frac{\dfrac{\partial u(x+\Delta x,t)}{\partial x}-\dfrac{\partial u(x,t)}{\partial x}}{\Delta x}=m\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

By imposing the condition that the function of the lateral deflection u(x,t) of the vibrating cable varies (changes) its magnitudes continuously along the cable length in the x-coordinate, i.e. $\Delta x \to 0$, and the increment of u(x,t), i.e. Δu is small enough to be neglected (i.e. $\Delta u \to 0$), the above expression may be expressed in the following form:

$$\lim_{\Delta x\to 0}\frac{\dfrac{\partial u(x+\Delta x,t)}{\partial x}-\dfrac{\partial u(x,t)}{\partial x}}{\Delta x}=\frac{\partial^2 u(x,t)}{\partial x^2}$$

or

$$P\frac{\partial^2 u(x,t)}{\partial x^2}=m\frac{\partial^2}{\partial t^2}\left(u+\frac{\Delta u}{2}\right)$$

We thus have the PDE for the free vibration analysis of long flexible cable in the form of:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=a^2\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (9.54)$$

where $a=\sqrt{\dfrac{P}{m}}$ with P = tension in the string with a unit of Newton (N) and m = mass of the cable per unit length in kg/m. The unit for the constant a in Equation (9.54) is thus **m/s**.

9.7.2 Solution of PDE for free vibration analysis of cable structures (p.318)

We will demonstrate the application of Equation (9.54) for the free vibration analysis of a long cable structure illustrated in Figure 9.18.

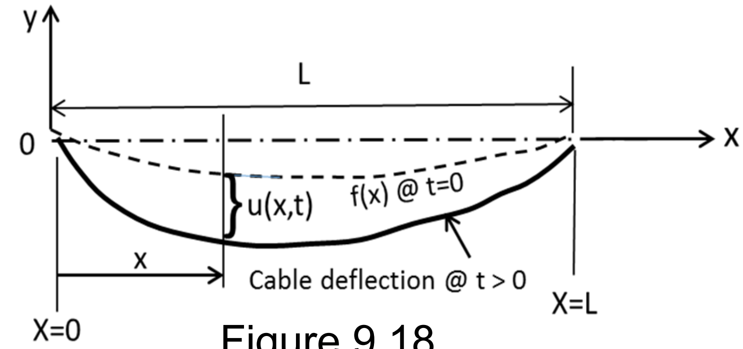


Figure 9.18

The cable initially has the shape in the dotted curve in Figure 9.18 that can be described by function f(x).

Lateral vibration of the cable with instantaneous magnitudes u(x,t) is induced to the cable by a small instantaneous disturbance with a slight vertical push to the cable downward that produces the instantaneous shape of the cable as shown in the sloid curve in the same figure at time t.

The free vibration of the cable with the lateral amplitudes u(x,t) is sustained by the “mass” of the cable material and its inherit “elasticity” of the cable. Our analysis is to solve u(x,t) for the physical situation described above.

We will use Equation (9.54) to solve for the u(x,t) by the separation of variables technique, as we did in Section 9.6 for heat conduction analysis. We will thus have the following mathematical model for the solution:

The PDE:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (9.54)$$

The initial conditions:

$$u(x,t)\big|_{t=0} = u(x,0) = f(x) \qquad (9.55a)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \dot{u}(x,0) = 0 \qquad (9.55b)$$

The end (boundary) conditions:

$$u(x,t)\big|_{x=0} = u(0,t) = 0 \qquad (9.56a)$$

$$u(x,t)\big|_{x=L} = u(L,t) = 0 \qquad (9.56b)$$

The partial differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{9.54}$$

The initial conditions:

$$u(x,t)\big|_{t=0} = u(x,0) = f(x) \tag{9.55a}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \dot{u}(x,0) = 0 \tag{9.55b}$$

The end (boundary) conditions:

$$u(x,t)\big|_{x=0} = u(0,t) = 0 \tag{9.56a}$$

$$u(x,t)\big|_{x=L} = u(L,t) = 0 \tag{9.56b}$$

Solution of Partial Differential Equation (9.54) by Separation of Variables Method (p.319):

We will need to separate these two variables x and t from the function u(x,t) in Equation (9.54) by letting:

$$u(x,t) = X(x)\,T(t) \tag{9.57}$$

The relation in Eq. (9.57) leads to:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[X(x)T(t)\right] = T(t)\frac{\partial X(x)}{\partial x} = T((t)X'(x)$$ and $$\frac{\partial (x,t)}{\partial t} = \frac{\partial}{\partial t}\left[X(x)T(t)\right] = X(x)\frac{\partial T(t)}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u(x,t)}{\partial x}\right] = T(t)X''(x)$$ and $$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial u(x,t)}{\partial t}\right] = T''(t)X(x)$$

Substituting the above expressions into Equation (9.54) will lead to:

LHS = $\frac{1}{a^2 T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$ = RHS = a constant ($-\beta^2$)

We thus have:

$$\frac{1}{a^2 T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta^2 \tag{9.58}$$

We will thus get two ordinary differential equations from (9.58):

$$\frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta^2$$

LHS = $-\beta^2$:

$$\frac{d^2T(t)}{dt^2} + a^2\beta^2\,T(t) = 0 \qquad (9.59)$$

RHS = $-\beta^2$:

$$\frac{d^2X(x)}{dx^2} + \beta^2\,X(x) = 0 \qquad (9.61)$$

After applying the same separation of variables as illustrated in Eq. (9.57) on the specified conditions in Equations (9.55) and (9.56), we get the two sets of ODEs with specific conditions in the following expressions:

$$\frac{d^2T(t)}{dt^2} + a^2\beta^2\,T(t) = 0 \qquad (9.59)$$

T(0) = f(x) (9.60a)

$$\left.\frac{dT(t)}{dt}\right|_{t=0} = 0 \qquad (9.60b)$$

$$\frac{d^2X(x)}{dx^2} + \beta^2\,X(x) = 0 \qquad (9.61)$$

X(0) = 0 (9.62a)

X(L) = 0 (9.62b)

Both Equations (9.59) and (9.61) are linear 2nd order ODEs with their solutions to be in the following forms:

T(t) = A Sin(βat) + B Cos(βat) (9.63)

X(x) = C Sin(βx) + D Cos(βx) (9.64)

The lateral amplitude of vibration cable u(x,t) in Figure 9.18 or the solution of Equation (9.54) can thus be expressed by sustituting the expressions in Equations (9.63) and (9.64) in Equation (9.57) to give:

$$T(t) = A\,\mathrm{Sin}(\beta at) + B\,\mathrm{Cos}(\beta at) \qquad X(x) = C\,\mathrm{Sin}(\beta x) + D\,\mathrm{Cos}(\beta x)$$

$$u(x,t) = [A\,\mathrm{Sin}(\beta at) + B\,\mathrm{Cos}(\beta at)][C\,\mathrm{Sin}(\beta x) + D\,\mathrm{Cos}(\beta x)]$$

where A, B, C, and D are arbitrary constants need to be determined from the given initial and boundary conditions given in Eqs. (9.60a,b) and (9.62a,b)

Determination of arbitrary constants:

Let us start with the solution: $X(x) = C\,\mathrm{Sin}(\beta x) + D\,\mathrm{Cos}(\beta x)$ in Eq. (9.64):

From Eq. (9.62a): $X(0) = 0$: $\longrightarrow$

$C\,\mathrm{Sin}(\beta{*}0) + D\,\mathrm{Cos}(\beta{*}0) = 0$, which means that $D = 0 \longrightarrow X(x) = C\,\mathrm{Sin}(\beta x)$

Now, from Eq. (9.62b): $X(L) = 0$: $\longrightarrow$ $X(L) = 0 = C\,\mathrm{Sin}(\beta L)$

At this point, we have the choices of letting C = 0, or Sin (βL) = 0 from the above relationship. A careful look at these choices will conclude that C ≠ 0 (why?), we thus have:

$\mathrm{Sin}(\beta L) = 0$

The above expression is a transcendental equation with an infinite number of roots for the solutions with βL= 0, π, 2π, 3π, 4π, 5π…………nπ , in which n is an integer number. We may thus obtain the values of the "separation constant, β" to be:

$$\longrightarrow \quad \beta_n = \frac{n\pi}{L} \qquad (n = 0,1,2,3,4\ldots\ldots\ldots..) \qquad (9.66)$$

Now, if we substitute the solution of X(x) in Eq. (9.64) with D=0 and $\beta_n = n\pi/L$ with n = 1, 2, 3,.. into the solution of u(x,t) expressed in the following form:

$$u(x,t) = [A\,Sin(\beta at) + B\,Cos(\beta at)][C\,Sin(\beta x) + D\,Cos(\beta x)]$$

We will get:

$$u(x,t) = \left(A Sin\frac{n\pi}{L}at + B Cos\frac{n\pi}{L}at\right) C\,Sin\frac{n\pi}{L}x \qquad \text{(n = 1, 2, 3,,……..)}$$

By combining constants A, B and C in the above expression, we have the interim solution of u(x,t) to be:

$$u(x,t) = \left(a_n\ Sin\frac{n\pi}{L}at\ +\ b_n\ Cos\frac{n\pi}{L}at\right) Sin\frac{n\pi}{L}x \qquad \text{(n = 1, 2, 3,,……..)}$$

We are now ready to use the two initial conditions in Eqs (9.55.a) and (9.55b) to determine constants a_n and b_n in the above expression:

Let us first look at the condition in Eq. (9.55b): $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 \longrightarrow$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 = \frac{n\pi a}{L}\left.\left(a_n Cos\frac{n\pi at}{L} - b_n Sin\frac{n\pi at}{L}\right)\right|_{t=0} Sin\frac{n\pi}{L}x$$

But since $Sin\frac{n\pi}{L}x \neq 0$ (why?) $\longrightarrow$ $a_n = 0$ $\longrightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} b_n\,Cos\frac{n\pi a}{L}t\ Sin\frac{n\pi a}{L}x$$

Thus, the only remaining constants to be determined are: b_n in the above expression.

Determination of constant coefficients b_n in the following expression (p.321):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, Cos \frac{n\pi a}{L} t \; Sin \frac{n\pi a}{L} x$$

The last remaining condition of **u(x,o) = f(x)** in Equation (9.55a) will be used for this purpose, in which f(x) is the given initial shape of the string.

Thus, by letting u(x,0) = f(x), we will have:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \; Sin \frac{n\pi x}{L} = f(x) \qquad \text{with } 0 \le x \le L$$

There are a number of ways to determine the coefficients b_n in the above expression. What we will do is to follow the orthogonality of trigonometric functions in Section 9.5.1 (p.302) to determine the coefficient b_n in the following way:

$$b_n = \frac{2}{L} \int_0^L f(x) \; Sin \frac{n\pi x}{L} \, dx \tag{9.68}$$

The complete solution of the amplitude of lateral vibrating string u(x,t) becomes:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L} \left(\int_0^L f(x) \, Sin \frac{n\pi x}{L} dx \right) Cos \frac{n\pi at}{L} \, Sin \frac{n\pi x}{L} \tag{9.69}$$

9.7.3 Convergence of Series Solutions (p.322)

Solution to partial differential equations by the separation of variables technique such as presented in Sections 9.5 to 9.7 include summations of infinite number of terms associated with the infinite number of roots of transcendental equation (or characteristic equations as mentioned in Chapter 4.The solution in Equation (9.69) for the PDE in Equation (9.54) is also in the form of infinite series.

Numerical solutions of these equations can be obtained by summing up the solutions with each assigned value of n, that is with n = 1, 2, 3, …….to a very large integer number.

In normal circumstances, these infinite series solutions should converge fairly rapidly, so one needs only to sum up approximately a dozen terms with the number n up to 12 for reasonably accurate solutions of the problems.. However, the effect of the convergences of infinite series, such as the one in Equation (9.69) on the accuracy of the analytical results remains a concern to engineers in their analyses.

We will demonstrate the convergence of a series solution related to Equation (9.69) for the vibration of a long cable similar to the situation depicted in Figure 9.18 with L = 20 m and the constant coefficient a = 120 m/s. We assume that the initial shape of the cable can be described by a function $f(x) = 0.25 \sin \frac{\pi}{L} x$

The magnitude of the amplitude of vibrating cable at x = 5 m at t = 1 second is from Equation (9.69) is of the form:

$$u(5,1) = \frac{1}{40} \sum_{n=1}^{\infty} \left[\cos(6n\pi) \sin \frac{n\pi}{4} \left(\int_0^{20} \sin \frac{n\pi}{20} \right) x dx \right]$$

or in the form with numerical values of n = 1,2,3,….,n:

$$u(5,1) = u_1 + u_2 + u_3 + u_4 + u_5 + \ldots + u_n$$

We used the MicroSoft Excel software to compute the numerical values of u(5,1) with n = 1,2,3,…,16 with the computed results shown in the following Table:

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
1.6 E-2	3.8 E-2	2.8 E-2	0	-9.38 E-3	3.22 E-3	9.38 E-3	0	-5.63 E-3	1.14 E-3	5.63 E-3	0	-4.02 E-3	5.77 E-4	4.02 E-3	0

and with more terms with additional values of n (up n = 30) in Figure 9.19:

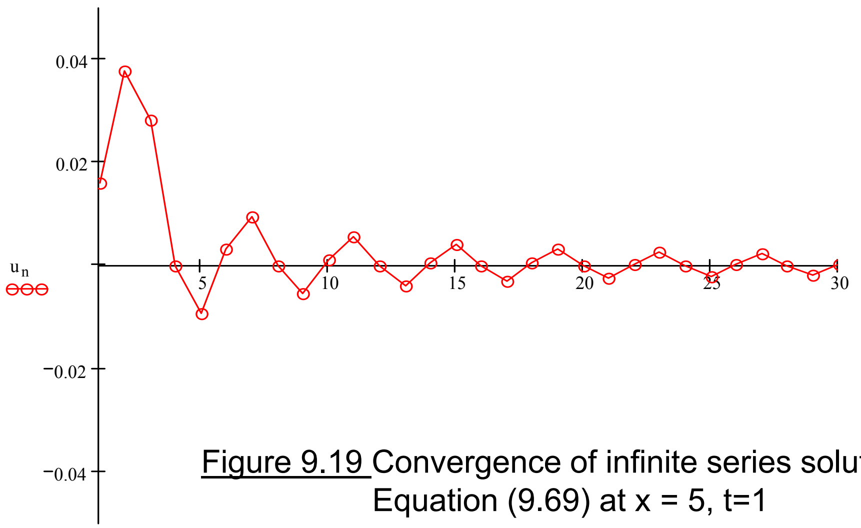
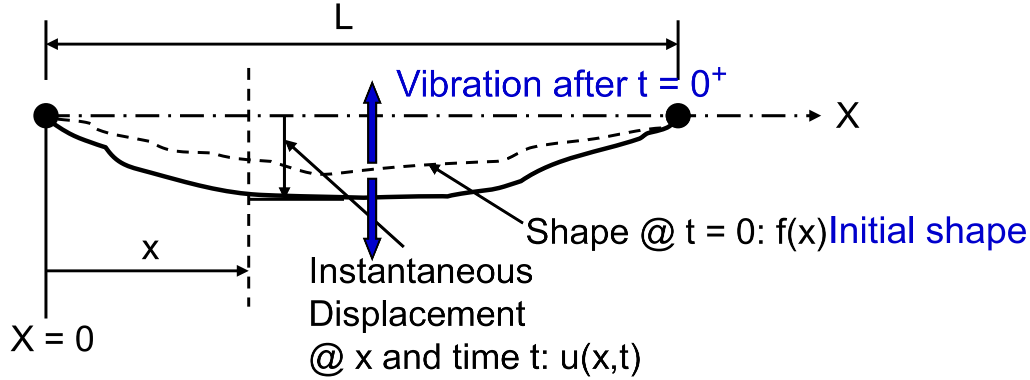


Figure 9.19 Convergence of infinite series solution of Equation (9.69) at x = 5, t=1

We observed from this particular case of numerical solutions of the infinite series solution of Equation (9,69) that inclusion of the first 20 terms in the series (i.e., n = 1,2,3,…..,20) would offer reasonably accurate solution of u(5,1) because of the continuous diminishing of the effects of the values of u(5,1) with the inclusion of terms with additional terms with n-values, as illustrated in this figure.

9.7.4 Modes of Vibration of Cable Structures (p.323)



We have just derived the solution on the AMPLITUDES of vibrating cables, u(x,t) to be:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{L}\left(\int_0^L f(x)\,Sin\frac{n\pi x}{L}dx\right)Cos\frac{n\pi at}{L}\,Sin\frac{n\pi x}{L} \qquad (9.69)$$

We realize from the above expression that the solution consists of INFINITE number of terms with n = 1, n = 2, n = 3,……… What it means is that each term alone in the infinite series in Equation (9.69) is a VALID solution. Hence: u(x,t) with one term with n = 1 only is one possible solution, and u(x,t) with n = 2 only is another possible solution, and so on and so forth.

Consequently, because the solution u(x,t) also represents the INSTANTANEOUS SHAPE of the vibrating string, there could be many POSSIBLE instantaneous shape of the vibrating string depending on what the terms in Eq. (9.69) are used.

Predicting the possible forms (or INSTATANEOUS SHAPES) of a vibrating string is called **MODAL ANALYSIS**

The First Three Modes of Vibrating Cables:

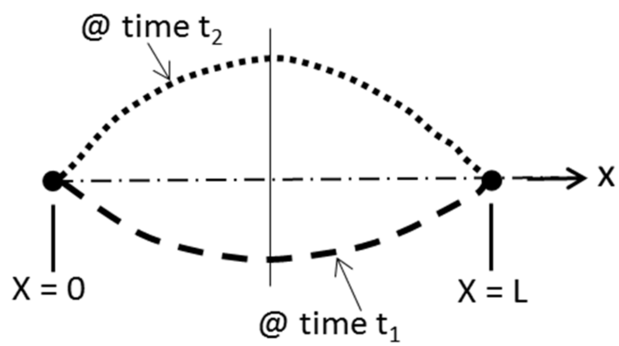
We will use the solution in Eq, (9.69) to derive the first three modes of a vibrating string.

Mode 1 with n = 1 in Eq. (9.69):

$$u_1(x,t) = \left(b_1 \, Cos \frac{\pi at}{L} \right) Sin \frac{n\pi}{L} x \qquad (9.69)$$

The SHAPE of the Mode 1 vibrating string can be illustrated according to Eq, (91.71a) as:

Figure 9.20



We observe that the maximum amplitudes of vibration occur at the mid-span of the string, As illustrated in Figure 9.20.

The corresponding frequency of vibration is obtained from the coefficient in the argument of the cosine function with time t in Equation (9.69), i.e.:

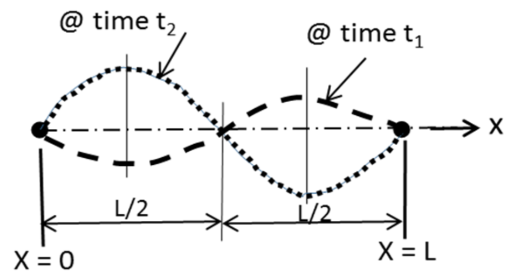
$$f_1 = \frac{\pi a / L}{2\pi} = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{P}{m}} \qquad (9.70)$$

where P = tension in Newton or pounds, and m = mass density of string/unit length in kg/m^3 or slugs/in.

Mode 2 with n = 2 in Eq. (9.69):
we will have the amplitude of the vibrating cable to be:

$$u_2(x,t) = \left(b_2 Cos\frac{2\pi a}{L}t\right) Sin\frac{2\pi}{L}x \qquad (9.71a)$$

Possible shape of the cable in Mode 2 vibration:

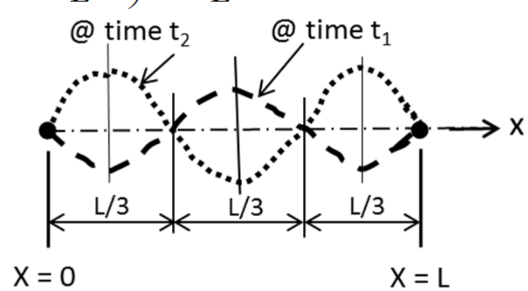


Frequency of Mode 2 vibration: $$f_2 = \frac{2\pi a/L}{2\pi} = \frac{a}{L} = \frac{1}{L}\sqrt{\frac{P}{m}} \qquad (9.72)$$

Mode 3 with n = 3 in Eq. (9.69):

$$u_3(x,t) = \left(b_3 Cos\frac{3\pi a}{L}t\right) Sin\frac{3\pi}{L}x \qquad (9.73a)$$

Possible shape of the Cable in Mode 3 vibration:



Frequency of Mode 3 vibration: $$f_3 = \frac{3\pi a/L}{2\pi} = \frac{3a}{2L} = \frac{3}{2L}\sqrt{\frac{P}{m}} \qquad (9.74)$$

Physical Importance of Modal Analysis in Vibration of Cable Structures

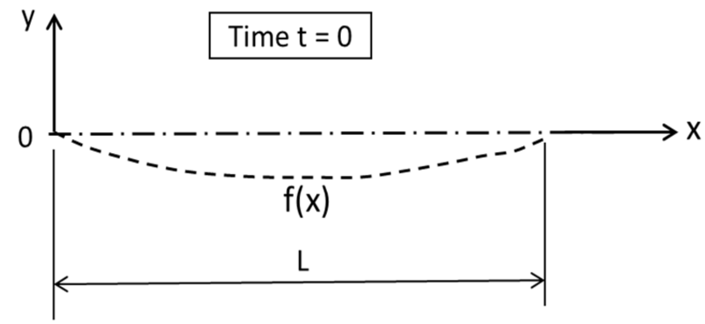
Modal analysis provides engineers with critical information on where the possible maximum amplitudes may exist when the string vibrates, and the corresponding frequency of occurrence.

Identification of locations of maximum amplitude allows engineers to predict possible locations of structural failure, and thus the vulnerable location of string (long cable) structures.

Of course, the multiple number of natural frequencies f_n such as indicated in the Equations (9.70), (9.72) and (9.74) for the cable in Figure 9.18 with Mode number n =1,2,3, are the indicators of what the frequencies of the applied intermittent loads should be avoided to this kind of structures in order to avoid the devastating resonant vibration of the structure. Modal analysis of cable structures such as illustrated in Figures 9.12-9.14 is thus a critically important part of the analysis.

Example 9.6 – A numerical case illustration of vibration analysis of a cable structure (p.325).

A flexible cable 10 m long is fixed at both ends with a tension of 500 N in the free-hung state (see the figure in the right.



The cable has a diameter of 1 cm and with a mass density ρ = 2.7 g/cm^3.

If the cable begins to vibrate by an instantaneous but small disturbance from its initial shape that can be described by the function $f(x) = 0.005x(1-x/10)$. Determine the following:

a) The applicable differential equation for the amplitudes of vibration of the cable represented by $u(x,t)$ in meters, in which t is the time into the vibration with a unit of second (s),

b) The mathematical expressions of the applicable initial and end conditions

c) The solution of $u(x,t)$ of the differential equation in meters

d) The solution of amplitude of the vibrating cable in Mode 1, i.e., $u_1(x,t)$ with the magnitude and location of the maximum deflection of the cable in this mode of vibration.

e) The numerical values of the frequencies of the first and second mode of vibration

f) The physical significance of these mode shapes.

Example 9.6 – A numerical case illustration of vibration analysis of a cable structure-Cont'd.

Solution (p.326):

We realize the following specific conditions:

The length of the cable L = 10 m, with a diameter d = 1 cm = 0.01 m
The cable is made of aluminum with a mass density ρ = 2.7 g/cm^3
The cable is subjected to a tension P = 500 N and with initial sag described by the function f(x): $f(x) = 0.005x\left(1 - \frac{x}{10}\right)$

a) The applicable differential equation for the amplitudes of vibration is Equation (9.54)

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (9.54)$$

with the constant coefficient a in the above equation determined by the following expression:

$a = \sqrt{\frac{P}{m}}$ in which P = tension in the cable = 500 N and m = mass per unit length which needs to be computed with given conditions. The mass per unit length of the cable is m = M/L where M = total mass of the cable with M = ρV with V being the volume of the cable.

We will get the volume of the cable be computed by the expression $V = \frac{\pi d^2}{4} L = 7.85x10^{-4}\, m^3$

We will thus have the total mass of the cable M = ρV = (2.7x10^3)(7.85x10^{-4}) = 2.12 kg, leading to the mass per unit length of the cable to be 0.212 kg/m. The constant coefficient according to the expression in Equation (9.54) is: $a = \sqrt{\frac{P}{m}} = \sqrt{\frac{500}{0.212}} = 48.56\, m/s$

The applicable PDE in equation (9.54) thus takes the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 2358.07 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (a)$$

Example 9.6 – A numerical case illustration of vibration analysis of a cable structure-Cont'd.

Solution – Cont'd (p.326)

b) The mathematical expressions of given initial and end conditions:

The initial conditions:

$$u(x,t)\big|_{t=0} = u(x,0) = f(x) = 0.005x\left(1-\frac{x}{10}\right) \tag{b1}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \dot{u}(x,0) = 0 \tag{b2}$$

The end conditions:

$$u(x,t)\big|_{x=0} = u(0,t) = 0 \tag{c1}$$

$$u(x,t)\big|_{x=L} = u(L,t) = u(10,t) = 0 \tag{c2}$$

c) The solution of u(x,t) of Equation (a) satisfying the given conditions in Equations (b1, b2) and Equations (c1 and c2) will be obtained as follows:

The solution of Equation (a) is similar to that of Equation (9.69) with a = 48.56 m/s in the following expression:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{10}\left\{\left[\int_0^{10}0.005x\left(1-\frac{x}{10}\right)\sin\frac{n\pi x}{10}dx\right]\left[\cos\frac{n\pi(48.56)}{10}t\right]\left[\sin\frac{n\pi}{10}x\right]\right\}$$

or

$$u(x,t)=\sum_{n=1}^{\infty}\frac{1}{5}\left\{\left[\int_0^{10}0.005x\left(1-\frac{x}{10}\right)\sin\frac{n\pi x}{10}dx\right][\cos 15.25nt][\sin 0.314nx]\right\} \tag{d}$$

or

$$u(x,t) = -0.1\sum_{n=1}^{\infty}\frac{1}{n\pi}\left(1+\frac{2-n^2\pi^2}{n^3\pi^3}\right)(\cos 15.25nt)(\sin 0.314nx) \tag{e}$$

Example 9.6 – A numerical case illustration of vibration analysis of a cable structure-Cont'd.

Solution – Cont'd (p.322)

d) The amplitude of the vibrating cable in Mode 1, i.e., $u_1(x,t)$ with the magnitude and location of the maximum deflection of the cable in this mode of vibration:

The required solution is obtained by letting n = 1 in Equation (e) as:

$$u_1(x,t) = -0.1\left[\frac{1}{\pi}\left(1+\frac{2-\pi}{\pi^3}\right)\right]\cos 15.25t \sin 0.314x = -0.02376\cos 15.25t \sin 0.314x \quad \text{(f)}$$

The maximum amplitude occurs at the mid-span of the cable at x = 5 m, and at the time when cos15.25t = 1.0. We thus have the maximum amplitude $u_{1,max}$ = 0.02376 m, or 2.376 cm at x = 5 m and at time 15.25t = π, or time t = π/15.25 = 0.2 s.

e) The numerical values of the frequencies of the first and second mode of vibration:

We may use Equations (9.70) and (9.72) to compute the numerical values of the frequencies of the first and second mode of vibration as follows:

$$f_1 = \frac{1}{2L}\sqrt{\frac{P}{m}} = \frac{1}{2x10}\sqrt{\frac{500}{0.212}} = 2.43\,Hz$$ for Mode 1, and $$f_2 = \frac{1}{L}\sqrt{\frac{P}{m}} = \frac{1}{10}\sqrt{\frac{500}{0.212}} = 4.86\,Hz$$ for Mode 2

f) The physical significance of these mode shapes to the design engineer

Engineers will use the outcomes of the above modal analysis to advise the users of this cable structure on possibility of devastating resonant vibration of the cable structure should the frequency of applied cyclic force, such as wind force coincides any of the natural frequencies computed in Part (e) in the solutions. The users will also be made aware of the locations where maximum amplitudes of vibration may occur as the mode shapes indicate in the modal analysis. They should avoid placing delicate attachments to these locations on the cable structure to avoid potential damages due to excessive vibration at these locations.

9.8 Partial Differential Equations for Transverse Vibration of Membranes (p.328)

Solids of plane geometry, such as thin plates are common appearance in machines and structures. Thin plates (or thin diaphragms) can be as small as printed electric circuit boards with micrometers in size or as large as floors in building structures. Like flexible cables, thin flexible plates are normally flexible and be vulnerable to transverse vibration. In some cases, these plates may rupture due to resonant vibrations, resulting in significant loss of property, and even human lives.

This section will derive appropriate PDEs that allow engineers to assess the amplitudes in free vibration of thin plates that are flexible enough to be simulated to thin membranes. Engineers may use this mathematical model for their modal analysis for the safe design of these types of machine components and structures.

9.8.1 **Derivation of partial differential equation for plate vibrations** (p.328)

We will derive the mathematical model for the transverse vibration of thin plates with the following idealizations and hypotheses:

1) The derivation of mathematical expressions is based on the lateral (vertical) displacement of solids of plane geometry that are flexible and offer no resistance to bending. In reality, the structure fits the description of “membranes“ in the subsequent analysis.

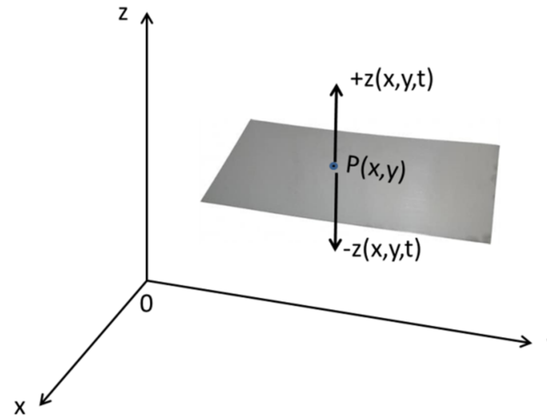


Figure 9.23 Transverse vibration of thin plate

2) Thin plates with unsupported large plane areas that are sufficiently flexible in lateral deformations.
3) Being flexible, there is no shear stress in the deformed thin plates.
4) The thin plate is initially flat with its edges fixed. There is an initial sag represented by a function f(x,y) sustained by in-plane tension P per unit length of the plate in all directions. The tension P is large enough to justify neglecting the weight of the plate.
5) Figure 9.23 defines the plate in the (x,y) plane with lateral displacement z(x,y,t), the amplitude of vibration of the plate at the locations defined by the x-y coordinates and at time t.
6) Every part of the plate vibrates in the direction perpendicular to the plane surface of the plate, i.e., in the z-coordinate as illustrated in Figure 9.24 The slopes of the deformed surface of the plate at all edges are small.
7) The mass per unit area of the plate, designated by the symbol (m) is uniform throughout the plate.

We notice that the solution of the amplitudes of vibrating membrane (or thin plate) z(x,y,t) now involves 3 independent variables: x,y and t. We may well image that it would be a much more complicated analysis problem than the cases that we have covered so far in this book.

9.8.1 **Derivation of partial differential equation for plate vibrations –** cont'd (p.329)

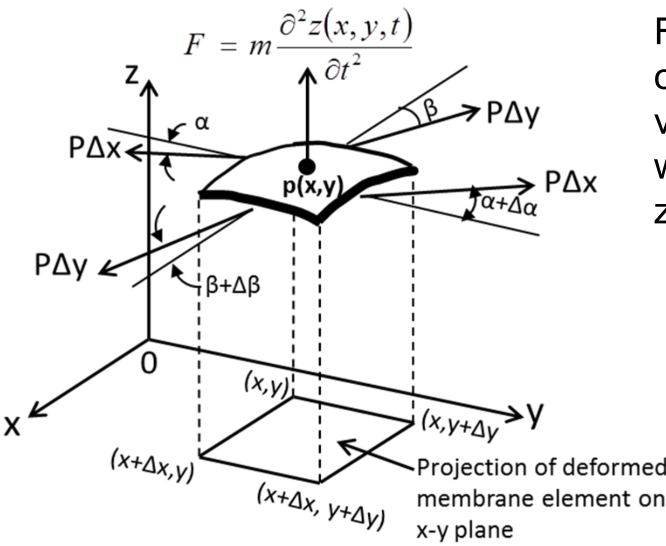


Figure 9.24 A free-body diagram of forces in an element of vibrating membrane at time t

Figure 9.24 is a free-body diagram which shows all forces acting on a small deformed element of the plate during a lateral vibration. The situation satisfies a dynamic equilibrium condition with the summation of all forces present at time t be equal to zero. Mathematically we may express this condition in the form:

$$\sum F_z = 0$$

The induced dynamic force F by Newton's second law plays a major role in the formulation of the above equilibrium of forces. Mathematically, this force may be expressed as:

$$F = m\frac{\partial^2 z(x,y,t)}{\partial t^2}$$

From Figure 9.24, we have the following dynamic equilibrium conditions:

$$P\Delta x\sin(\alpha+\Delta\alpha) - P\Delta x\sin\alpha + P\Delta y\sin(\beta+\Delta\beta) - P\Delta y\sin\beta - m(\Delta x\Delta y)\frac{\partial^2 z[x,y,t]}{\partial t^2} = 0 \qquad (9.75)$$

where m = mass per unit area of the plate material.

Idealization No. 6 indicates that both angles α and β are small, leading to the following approximate relationships:

$$\sin\alpha \approx \tan\alpha = \frac{\partial z\left(x+\frac{\Delta x}{2}, y, t\right)}{\partial y} \qquad \sin(\alpha+\Delta\alpha) \approx \tan(\alpha+\Delta\alpha) = \frac{\partial z\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y}$$

$$\sin\beta \approx \tan\beta = \frac{\partial z\left(x, y+\frac{\Delta y}{2}, t\right)}{\partial x} \qquad \sin(\beta+\Delta\beta) \approx \tan(\beta+\Delta\beta) = \frac{\partial z\left(x+\Delta x, y+\frac{\Delta y}{2}, t\right)}{\partial x}$$

9.8.1 **Derivation of partial differential equation for plate vibrations –** cont'd

Substituting the above 4 approximate relationships into Equation (9.75) will result in the following expression:

$$P\Delta x\left[\frac{\partial z\left(x+\frac{\Delta x}{2},y+\Delta y,t\right)}{\partial y}-\frac{\partial z\left(x+\frac{\Delta x}{2},y,t\right)}{\partial y}\right]+P\Delta y\left[\frac{\partial z\left(x+\Delta x,y+\frac{\Delta y}{2},t\right)}{\partial x}-\frac{\partial z\left(x,y+\frac{\Delta y}{2},t\right)}{\partial x}\right]$$

$$-m\Delta x\Delta y\frac{\partial^2 z(x,y,t)}{\partial t^2}=0$$

The following expression is obtained by dividing the above expression by ΔxΔy:

$$P\left[\frac{\frac{\partial z\left(x+\frac{\Delta x}{2},y+\Delta y,t\right)}{\partial y}-\frac{\partial z\left(x+\frac{\Delta x}{2},y,t\right)}{\partial y}}{\Delta y}+\frac{\frac{\partial\left(x+\Delta x,y+\frac{\Delta y}{2},t\right)}{\partial x}-\frac{\partial z\left(x,y+\frac{\Delta y}{2},t\right)}{\partial x}}{\Delta x}\right]$$

$$-m\frac{\partial^2 z(x,y,t)}{\partial t^2}=0$$

Given that the lateral deformation of the plate continuously varying with the locations on the plane defined by the x- and y-coordinate, we should have the following relationships shown in the next slide.

9.8.1 **Derivation of partial differential equation for plate vibrations –** cont'd

$$\ell im_{\Delta y \to 0} \frac{\dfrac{\partial z\left(x+\dfrac{\Delta x}{2}, y+\Delta y, t\right)}{\partial y} - \dfrac{\partial z\left(x+\dfrac{\Delta x}{2}, y, t\right)}{\partial y}}{\Delta y} = \frac{\partial^2 z(x,y,t)}{\partial y^2}$$

and

$$\ell im_{\Delta x \to 0} \frac{\dfrac{\partial z\left(x+\Delta x, y+\dfrac{\Delta y}{2}, t\right)}{\partial x} - \dfrac{\partial z\left(x, y+\dfrac{\Delta y}{2}, t\right)}{\partial x}}{\Delta x} = \frac{\partial^2 z(x,y,t)}{\partial x^2}$$

The equilibrium equation in (9.75) thus has the following form with Δx→0 and Δy→0 for continuous variation of the amplitude of vibration of the plate in both x- and y-coordinates with:

$$P\left[\frac{\partial^2 z(x,y,t)}{\partial y^2} + \frac{\partial^2 z(x,y,t)}{\partial x^2}\right] - m\frac{\partial^2 z(x,y,t)}{\partial t^2} = 0$$

or in the form of:

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \qquad (9.76)$$

where the constant a in Equation (9.76) has the similar form as in Equation (9.54) but with different meaning:

$$a = \sqrt{\frac{P}{m}} \qquad (9.77)$$

where P is the tension per unit length with unit N/m, and m is the mass per unit area kg/m^2. The constant a thus has a unit of m/s.

9.8.2 Solution of a Partial Differential Equation for Thin Plate Vibration (p.331)

We will use Equation (9.76) to compute the magnitudes of a transverse vibrating thin plate such as a computer mouse pad, induced by a slight instantaneous disturbance in the z-direction in Figure 9.25.

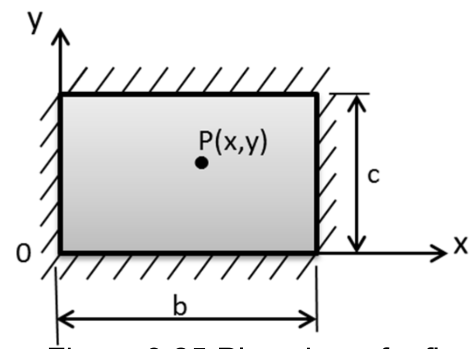


Figure 9.25 Plan view of a flexible thin plate undergoing a transverse vibration.

We will have the following PDE and the given appropriate initial and boundary conditions for the solution of the magnitudes of the vibrating plate at given time t, i.e. z(x,y,t) in Equation (9.76):

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \qquad (9.76)$$

A) The boundary conditions:

$$z(x,y,t)\big|_{x=0} = z(0,y,t) = 0 \qquad (a1)$$

$$z(x,y,t)\big|_{x=b} = z(b,y,t) = 0 \qquad (a2)$$

$$z(x,y,t\big|_{y=0} = z(x,0,t) = 0 \qquad (b1)$$

$$z(x,y,t)\big|_{y=c} = z(x,c,t) = 0 \qquad (b2)$$

B) The initial conditions:

$$z(x,y,t)\big|_{t=0} = f(x,y) \qquad (c1)$$

$$\left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y) \qquad (c2)$$

The function g(x,y) in Equation (c2) is another given function that describes the velocity of the plate across the plane of the plate at the inception of the vibration.

9.8.2 **Solution of a Partial Differential Equation for Thin Plate Vibration –** Cont'd

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \qquad (9.76)$$

A) The boundary conditions:

$$z(x,y,t)\big|_{x=0} = z(0,y,t) = 0 \qquad (a1)$$

$$z(x,y,t)\big|_{x=b} = z(b,y,t) = 0 \qquad (a2)$$

$$z(x,y,t)\big|_{y=0} = z(x,0,t) = 0 \qquad (b2)$$

$$z(x,y,t)\big|_{y=c} = z(x,c,t) = 0 \qquad (b1)$$

B) The initial conditions:

$$z(x,y,t)\big|_{t=0} = f(x,y) \qquad (c1)$$

$$\left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y) \qquad (c2)$$

We will use the separation of variables techniques to solve the above equations with the specified boundary and initial conditions. This technique requires the solution z(x,y,t) of Equation (9.76) to be the product of **3** separate functions each contains only one of the 3 independent variables as:

$$Z(x,y,t) = X(x)Y(y)T(t) \qquad (9.78)$$

Substituting the expression in Equation (9.78) into Equation (9,76) will lead to the following expression:

$$\text{LHS} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = \text{RHS}$$

The equality of both sides in the above express is possible if both sides equal to a constant by the principle of mathematics. We thus have the following valid expression instead:

$$\text{LHS} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = -\lambda^2 = \text{RHS} \qquad (d)$$

where λ in Equation (d) is the first separation constant in this analysis

Equation (d) results in the following 2 ordinary differential equations (ODE):

The first ordinary differential equation for function X(x):

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \qquad (e)$$

and another equality leads to the 2nd ODE:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} + \lambda^2 \qquad (f)$$

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \qquad (9.76)$$

A) The boundary conditions:

$$z(x,y,t)|_{x=0} = z(0,y,t) = 0 \qquad (a1) \qquad\qquad z(x,y,t)|_{x=b} = z(b,y,t) = 0 \qquad (a2)$$

$$z(x,y,t|_{y=0} = z(x,0,t) = 0 \qquad (b2) \qquad\qquad z(x,y,t)|_{y=c} = z(x,c,t) = 0 \qquad (b1)$$

B) The initial conditions:

$$z(x,y,t)|_{t=0} = f(x,y) \qquad (c1) \qquad\qquad \left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y) \qquad (c2)$$

For the same reason; the validity of the equality in Equation (f) requires that both sides of the equality to be a same constant as shown below:

$$\text{LHS} = \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} + \lambda^2 = \mu^2 = \text{RHS} \qquad (g)$$

where μ is the second separation constant in expression (g).

We may derive another two ordinary differential equations from the expression in Equation (g):

$$\frac{d^2Y(y)}{dy^2} + \mu^2 Y(y) = 0 \qquad (h)$$

$$\frac{d^2T(t)}{dt^2} + a^2\left(\lambda^2 + \mu^2\right)T(t) = 0 \qquad (j)$$

We have thus separate the variables in PDE in (9.76) onto 3 ODEs using the separation of variables technique in Equation (9.78) as shown below:

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \quad (9.76) \;\rightarrow$$

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \qquad \textbf{(e)}$$

$$\frac{d^2Y(y)}{dy^2} + \mu^2 Y(y) = 0 \qquad (h)$$

$$\frac{d^2T(t)}{dt^2} + a^2\left(\lambda^2 + \mu^2\right)T(t) = 0 \qquad \textbf{(j)}$$

9.8.2 **Solution of a Partial Differential Equation for Thin Plate Vibration** – Cont'd

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \quad (9.76)$$

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \quad (e)$$

$$\frac{d^2Y(y)}{dy^2} + \mu^2 Y(y) = 0 \quad (h)$$

$$\frac{d^2T(t)}{dt^2} + a^2\left(\lambda^2+\mu^2\right)T(t) = 0 \quad (j)$$

A) The boundary conditions:

$$z(x,y,t)\big|_{x=0} = z(0,y,t) = 0 \quad (a1) \qquad z(x,y,t)\big|_{x=b} = z(b,y,t) = 0 \quad (a2)$$

$$z(x,y,t)\big|_{y=0} = z(x,0,t) = 0 \quad (b1) \qquad z(x,y,t)\big|_{y=c} = z(x,c,t) = 0 \quad (b2)$$

B) The initial conditions:

$$z(x,y,t)\big|_{t=0} = f(x,y) \quad (c1) \qquad \left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y) \quad (c2)$$

We notice that all the 3 ordinary differential equations (ODE) in Equations (e), (h) and (j) are 2nd order linear ODEs. Solutions of these equations are available in Section 8.2 (p.243), as shown below:

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x \quad (k1)$$

$$Y(y) = c_3\cos\mu y + c_4\sin\mu y \quad (k2)$$

$$T(t) = c_5\cos a\sqrt{\lambda^2+\mu^2}\,t + c_6\sin a\sqrt{\lambda^2+\mu^2}\,t \quad (K3)$$

We may follow the similar procedures presented in Section 9.5.1 (p.298), 9.6.1 (p.308) and 9.7.2 (p.318) in determining he constants c_1, c_2, c_3 and c_4 in Equations (k1)and (k2), and we may determined the two separation constants λ and μ to be: $\lambda=m\pi/b$ with m = 1,2,3,…., and $\mu=n\pi/c$ with n = 1,2,3,…., respectively, as well as $c_1=c_3=0$ using the conditions in Equations(a1) and (a2). Consequently, we will have:

$$X(x) = c_2\sin\lambda x = c_{2,m}\sin\frac{m\pi x}{b} \quad with\ \ m = 1,2,3,\ldots\ldots\ldots \quad (m1)$$

$$Y(y) = c_4\sin\mu y = c_{4,n}\sin\frac{n\pi y}{c} \quad with\ \ n = 1,2,3,\ldots\ldots\ldots\ldots\ldots\ldots \quad (m2)$$

9.8.2 **Solution of a Partial Differential Equation for Thin Plate Vibration –** Cont'd

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \quad (9.76)$$

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \quad \text{(e)}$$

$$\frac{d^2Y(y)}{dy^2} + \mu^2 Y(y) = 0 \quad \text{(h)}$$

$$\frac{d^2T(t)}{dt^2} + a^2\left(\lambda^2 + \mu^2\right)T(t) = 0 \quad \text{(j)}$$

$$X(x) = c_2 \sin \lambda x = c_{2,m} \sin \frac{m\pi x}{b} \quad \text{with } m = 1, 2, 3, \ldots\ldots\ldots \quad \text{(m1)}$$

$$Y(y) = c_4 \sin \mu y = c_{4,n} \sin \frac{n\pi y}{c} \quad \text{with } n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(m2)}$$

The constant c_5 and c_6 involved with the solution in Equation (k3) may be determined with the initial conditions specified in Equations (c1) and (c2) for z(x,y,t):in the following way:

$$z(x,y,t) = X(x)Y(y)T(t)$$

$$= \left(c_{2,m}\sin\frac{m\pi x}{b}\right)\left(c_{4,n}\sin\frac{n\pi y}{c}\right)\left(c_5 \cos a\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}\, t + c_6 \sin a\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}\, t\right) \quad \text{(n)}$$

Now if we let: $\omega_{mn} = \sqrt{\lambda_m^2 + \mu_n^2} = \sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2}$ (p)

We may express the solution of PDE in Equation (9.76) in the following form:

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sin\frac{m\pi}{b}x\right)\left(\sin\frac{n\pi}{c}y\right)\left(A_{mn}\cos a\omega_{mn}t + B_{mn}\sin a\omega_{mn}t\right) \quad \textbf{(9.79)}$$

where the multi-valued constant coefficients A_{mn} and B_{mn} are determined by the two remaining initial conditions in Equations (c1) and (c2) with the forms:

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x,y)\left(\sin\frac{m\pi}{b}x\right)\left(\sin\frac{n\pi}{c}y\right)dxdy \quad \text{(9.80a)}$$

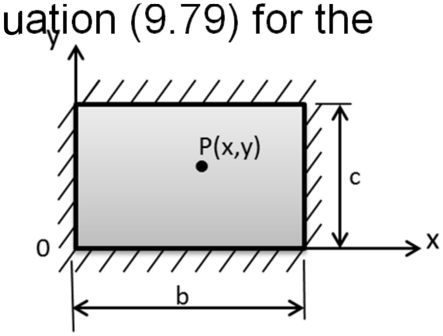
$$B_{mn} = \frac{4}{abc\omega_{mn}}\int_0^c\int_0^b g(x,y)\left(\sin\frac{m\pi}{b}x\right)\left(\sin\frac{n\pi}{c}y\right)dxdy \quad \text{(9.80b)}$$

with m,n = 1,2,3,….

9.8.3 **Numerical solution of the Partial Differential Equation for thin plate vibration** (p.334)

Numerical solution of the amplitudes of transverse vibration of flexible plates given in Equation (9.79) with coefficients in Equations (9.80a) and (9.80b) is a much more tedious and complicated than one would imagine. However, numerical solution can offer engineers much needed perception on the natural frequencies of the plate structures – much more so than what we may observe from the analytical solutions that we may obtain from the aforementioned math expressions in the aforementioned equations.

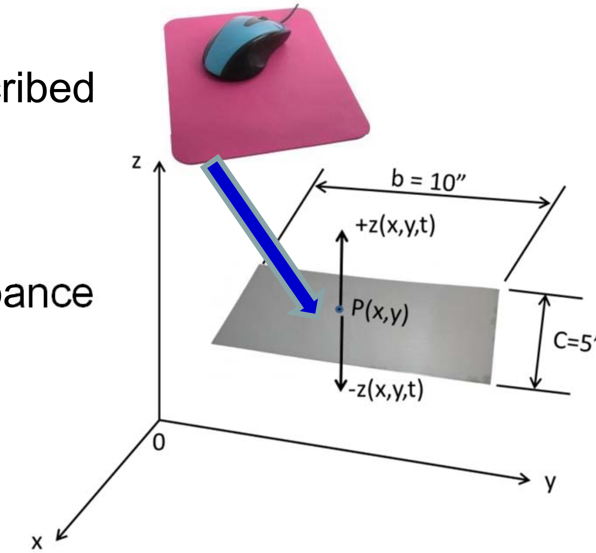
What we will present in this section is the numerical solution of Equation (9.79) for the shapes of a thin flexible plate (a computer mouse pad) illustrated in Figure 9.25 for its first three modes in vibration.



Dimensions of this plate is shown in lower figure in the right with the edges b=10” and c=5” and thickness of 0.185”. The pad is made of synthetic rubber, so it is flexible.

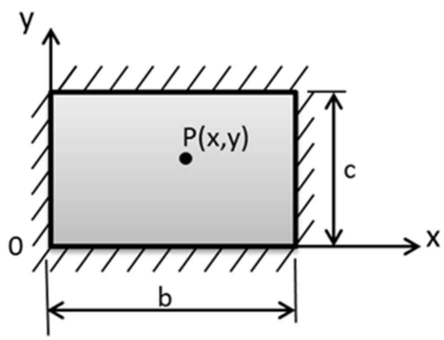
The pad has fixed edges with initial sagging that can be described by a function $f(x,y) = (10-x)(5-y)$ with an in-plane tension, $P = 0.5\ lb_f/in$

Vibration of the pad induced by a slight instantaneous disturbance lateral to the pad from a static equilibrium condition (i.e., zero velocity) with which $g(x,y) = 0$.in Equation (c2).



We will use Equation (9.76) to solve for the magnitudes of this thin plate:

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = a^2\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}\right] \qquad (9.76)$$



with the following boundary conditions:

$$z(x,y,t)\big|_{x=0} = z(0,y,t) = 0 \qquad (a1) \qquad z(x,y,t)\big|_{x=b} = z(b,y,t) = 0 \qquad (a2)$$

$$z(x,y,t)\big|_{y=0} = z(x,0,t) = 0 \qquad (b1) \qquad z(x,y,t)\big|_{y=c} = z(x,c,t) = 0 \qquad (b2)$$

and the initial conditions:

$$z(x,y,t)\big|_{t=0} = f(x,y) = (10-x)(5-y) \qquad (c1) \qquad \left.\frac{\partial z(x,y,t)}{\partial t}\right|_{t=0} = g(x,y)=0 \qquad (c2)$$

The constant coefficient "a" in the RHS of Equation (9.76) can be computed to be:

$$a = \sqrt{\frac{Pg}{\rho}} = \sqrt{\frac{(0.5\ lb_f/in)(32.2\ ft/s^2)}{0.00155\ lb_m/in^2}\left(\frac{12in}{ft}\right)} = 353.05\ in/s \qquad (d)$$

The frequency ω_{mn} required to compute the periods T is expressed in Equation (p) in Section 9.8.2 with eigenvalues $\lambda_m = m\pi/10$ and $\mu_n = n\pi/5$ with m, n = 1,2,3,……..as shown in the same Section.

The mode shapes of this plate from free lateral vibration analysis is computed by the following expression:

$$z(x,y,t) = A_{mn}\left(\sin\frac{m\pi x}{10}\right)\left(\sin\frac{n\pi y}{5}\right)\left(A_{mn}\cos a\omega_{mn}t\right) \qquad (9.81)$$

where

$$A_{mn} = \frac{16(bc)^2\left[1+(-1)^{n+1}\right]\left[1+(-1)^{m+1}\right]}{m^3n^3\pi^6} \qquad (9.82)$$

We realize that m, n = 1, 2, 3, ……………., and b = 10" and c=5" in both Equations (9.81) and (9.82).

9.8.3 **Numerical solution of the Partial Differential Equation for thin plate vibration-**Cont'd

- **Graphical solutions of the first three (3) modes of free vibration of the thin plate with m=n = 1,2 and 3:**

Modal analysis of plate vibration is a very important engineering analysis that relates to the safe design of this type of structures because many such structures are expected to survive cyclic load applications. Such situation is vulnerable to structural failures in resonant vibration should the frequency of the applied cyclic loads coincides with any natural frequencies of the plate found in the modal analysis. Solutions for these natural frequencies of plates of given geometry and material properties requires the solution of the shape of the deformed plates at various modes, and it will also provide engineers with possible shapes of the plate under each of these modes of vibration.

In-depth descriptions of resonant vibration and modal analysis of structures were presented in both Sections 8.7.2 and 8.9.

Natural frequencies of the plate illustrated in Figure 9.25 requires us to compute the amplitudes of the plate z(x,y,t) in Equation (9.81) given below, with specific conditions as presented in Equations (a1,a2, b1, b2, c1 and c2):

$$z(x,y,t) = A_{mn}\left(\sin\frac{m\pi x}{10}\right)\left(\sin\frac{n\pi y}{5}\right)\left(A_{mn}\cos a\omega_{mn}t\right) \qquad (9.81)$$

Where the coefficients:

$$A_{mn} = \frac{16(bc)^2\left[1+(-1)^{n+1}\right]\left[1+(-1)^{m+1}\right]}{m^3n^3\pi^6} \qquad (9.82)$$

And the natural frequencies:

$$\omega_{mn} = \sqrt{\lambda_m^2+\mu_n^2} = \sqrt{\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2} \qquad (9.83)$$

With: m, n = 1, 2, 3, ……………., and b = 10" and c=5" in the above expressions.

Readers are reminded that for plate vibration analysis, Mode 1 vibration is obtained with m=n=1, Mode2 vibration with m=n=2, and Mode 3 vibration with m=n=3 are used in the above formulations.

9.8.3 **Numerical solution of the Partial Differential Equation for thin plate vibration-**Cont'd

- **Natural frequencies of the first three (3) modes of free vibration of the thin plate with m=n = 1,2 and 3:**

We have obtained the expression of the amplitude of a plate, z(x,y,t) in a free-vibration analysis shown in Equation (9.81), from which we may get the natural frequencies of the plate from the coefficient in the argumen of the cosine function:"cos($a\omega_{mn}$)t" in Equation (9.81). Hence the natural frequencies of the plate are: $\boldsymbol{a\omega_{mn}}$, with m=1,2,3,….., and n = 1,2,3,……..

$$z(x,y,t) = A_{mn}\left(\sin\frac{m\pi x}{10}\right)\left(\sin\frac{n\pi y}{5}\right)\left(A_{mn}\cos a\omega_{mn}t\right) \qquad (9.81)$$

where

$$A_{mn} = \frac{16(bc)^2\left[1+(-1)^{n+1}\right]\left[1+(-1)^{m+1}\right]}{m^3n^3\pi^6} \qquad (9.82)$$

and

$$\omega_{mn} = \sqrt{\lambda_m^2+\mu_n^2} = \sqrt{\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2} \qquad (p)$$

$$a = \sqrt{\frac{Pg}{\rho}} = \sqrt{\frac{(0.5\,lb_f/in)(32.2\,ft/s^2)}{0.00155\,lb_m/in^2}\left(\frac{12in}{ft}\right)} = 353.05\,in/s \qquad (d)$$

We may compute the natural frequencies of the first 3 modes to be:

Mode 1 with m=n=1:

$$f_1 = a\omega_{11} = 353.05\sqrt{\left(\frac{\pi}{10}\right)^2+\left(\frac{\pi}{5}\right)^2} = 78.94\ Rad/s$$

Mode 2 with m=n=2:

$$f_2 = a\omega_{22} = 353.05\sqrt{\left(\frac{2\pi}{10}\right)^2+\left(\frac{2\pi}{5}\right)^2} = 157.88\ Rad/s$$

Mode 3 with m=n=3:

$$f_3 = a\omega_{33} = 353.05\sqrt{\left(\frac{3\pi}{10}\right)^2+\left(\frac{3\pi}{5}\right)^2} = 236.82\ Rad/s$$

9.8.3 **Numerical solution of the Partial Differential Equation for thin plate vibration-**Cont'd

- **Shapes of the vibrating thin plate at various modes:**

$$a = \sqrt{\frac{Pg}{\rho}} = \sqrt{\frac{(0.5\ lb_f/in)(32.2\ ft/s^2)}{0.00155\ lb_m/in^2}\left(\frac{12in}{ft}\right)} = 353.05\,in/s \qquad (9.81)$$

where

$$A_{mn} = \frac{16(bc)^2\left[1+(-1)^{n+1}\right]\left[1+(-1)^{m+1}\right]}{m^3n^3\pi^6} \qquad (9.82)$$

and

$$\omega_{mn} = \sqrt{\lambda_m^2+\mu_n^2} = \sqrt{\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2} \qquad (p)$$

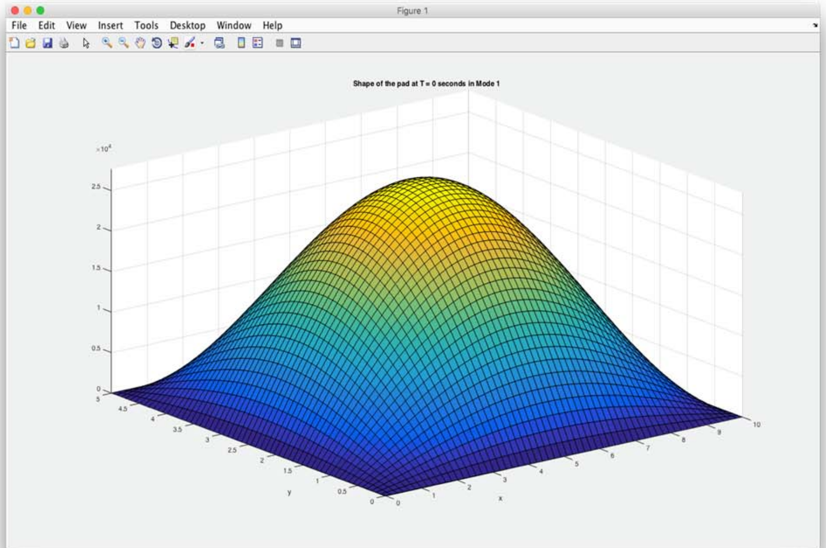
$$z(x,y,t) = A_{mn}\left(\sin\frac{m\pi x}{10}\right)\left(\sin\frac{n\pi y}{5}\right)\left(A_{mn}\cos a\omega_{mn}t\right) \qquad (d)$$

Modal shapes of thin plates require numerical solutions of z(x,y,t) of Equation (9.81) with m=n=1,2,3, which is a very tedious job. It will also be a great deal of laborious efforts to obtain graphical representations of these shapes. Consequently, we will use a commercially available MatLAB software (version R2015) available at the author's host university to perform these computations and present the computed modal shapes of the plates in graphs for the solutions.

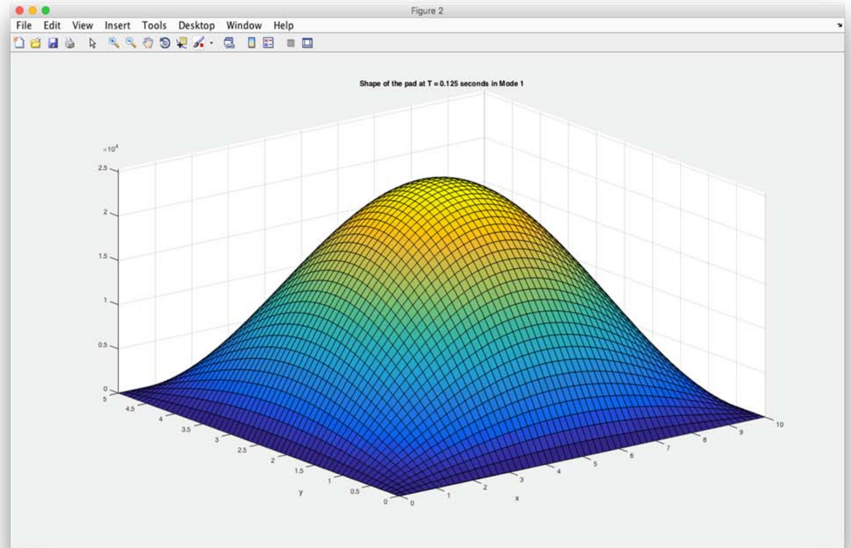
An overview of this software will be described in Section 10.5.2 of Chapter 10 (p.376), with inputs/output files for this analytical problem presented in Case 2 in Appendix 4 (P.473).

Graphical displays of the first two modal shapes of the thin plate with time t = 0, 1/8 and 1/4 seconds for Mode 1 (m=n=1) and Mode 2 with m=n=2 at t=1/8 and ¼ seconds will be shown in the next two slides.

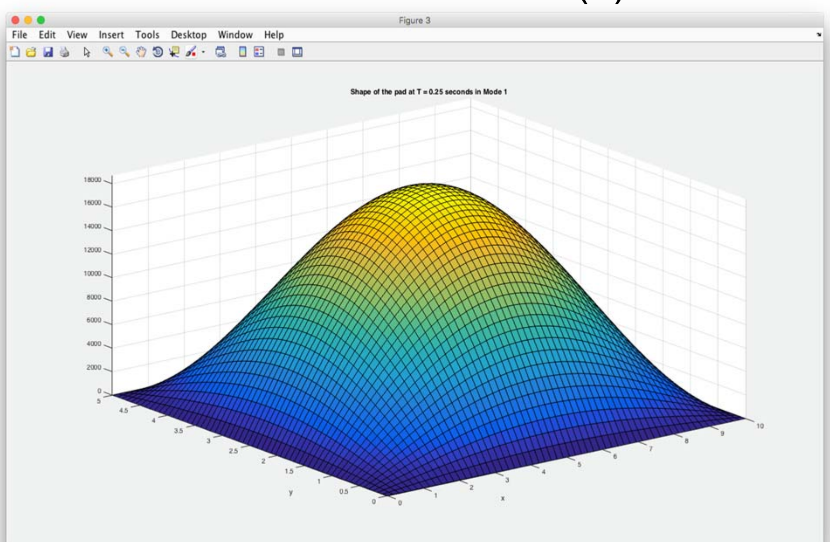
Modal **ONE** shapes of the thin plate at: (a) t=0, (b) t=1/8 second and (c) t=1/4 seconds



(a) At t=0, peak at ≈0.16”

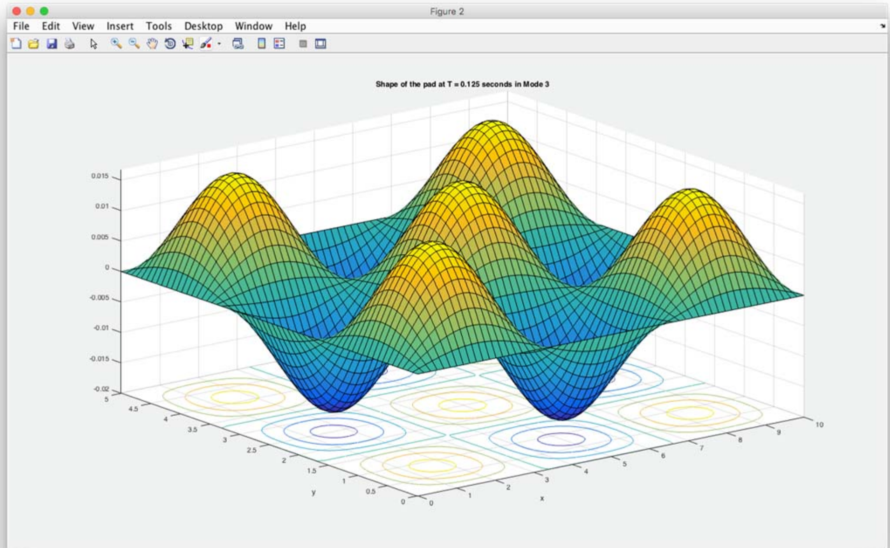


(b) At t=1/8 second, peak at ≈ 0.17 “

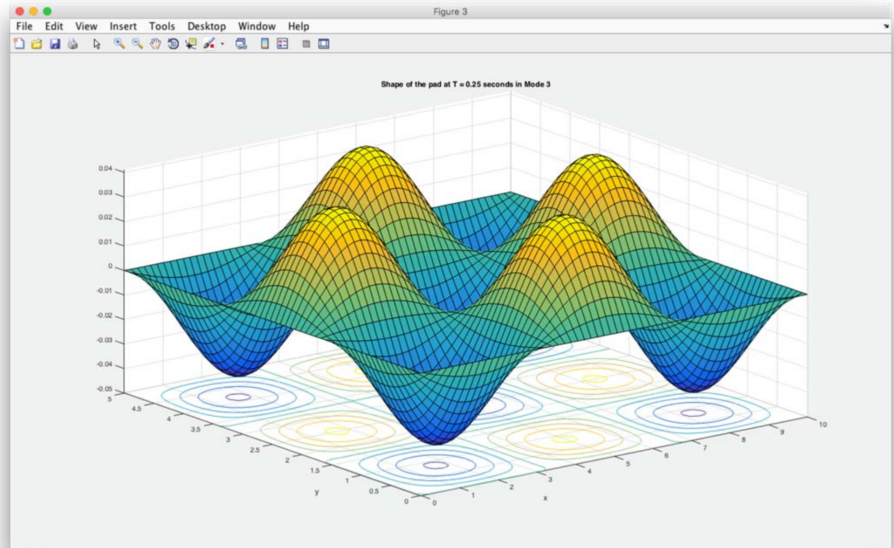


(c) At t = 1/4 seconds, Peak at≈0.14”

Modal **TWO** shapes of the thin plate at: (c) t=1/8 second and (d) t=1/4 seconds



(c) At t=1/8 second, peaks at 0.0125 “



(d) At t=1/4 second, peaks at 0.03 “

These modal shapes provide engineers with possible shape changes of the plate in vibrations. The illustrated shapes also indicate where the peak amplitudes of vibration of the flexible plate would occur, from which the design engineer should take precaution for not placing delicate attachments at these locations to avoid possible damages due to excessive deformation of the plate structure.

The computed natural frequencies with: f_1=78.94 rad/s, f_2=157.88 rad/s and f_3 = 236.82 rad/s will remind the potential users of this plate structure to avoid such frequencies when applying intermittent loads in order to avoid the devastating resonant vibration of this thin plate.